A uniqueness result for 2-soliton solutions of the KdV equation

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Abstract

Multisoliton solutions of the KdV equation satisfy nonlinear ordinary differential equations which are known as stationary equations for the KdV hierarchy, or sometimes as Lax-Novikov equations. An interesting feature of these equations, known since the 1970's, is that they can be explicitly integrated, by virtue of being finite-dimensional completely integrable Hamiltonian systems. Here we use the integration theory to investigate the question of whether the multisoliton solutions are the only nonsingular solutions of these ordinary differential equations which vanish at infinity. In particular we prove that this is indeed the case for 2-soliton solutions of the fourth-order stationary equation.

1   Introduction

The Korteweg-de Vries (or KdV) equation,

$$u_t = \frac{1}{4}(u_{xxx} + 6uu_x), \tag{1.1}$$

was first derived in the 1800's as a model for long, weakly nonlinear one-dimensional water waves ([23], see also equation (283 *bis*) on p. 360 of [4]). It was not until the 1960's, however, that the striking discovery was made that the equation has particle-like solutions known as solitons, whose interactions with each other are described by explicit multisoliton solutions [15, 18].

It is well-known that the profiles of multisoliton solutions, which are smooth functions that vanish rapidly at infinity, are critical points for variational problems associated with conserved functionals of KdV (see, e.g., [28]). By virtue of this property, the profiles are solutions of Lagrange multiplier equations, which take the form of nonlinear ordinary differential equations, sometimes known as

Lax-Novikov equations, or as the equations for stationary solutions of a family of time-dependent equations known as the KdV hierarchy (see below for details). In this paper we investigate the problem of establishing a converse to this statement: is it true that if a solution to a stationary equation for the KdV hierarchy is, together with enough of its derivatives, square integrable on the real line, then must it be a profile of a multisoliton solution?

For the case of the KdV equation itself (the first equation in the hierarchy), it is an elementary exercise to prove that the only stationary solutions in L^2 are the well-known solitary-wave solutions. Here we give a proof that the answer is also affirmative for the case of the fourth-order stationary equation for the second, fifth-order, equation in the KdV hierarchy (see Theorem 5.2 below). Much of our proof easily generalizes to the other stationary equations for the hierarchy, but some work remains to be done to complete the proof in the general case.

Our proof proceeds by integrating the stationary equations, using the method developed in the pioneering work of Dubrovin [10], Its and Matveev [20], Lax [24], and Novikov [29] on solutions of the periodic KdV equation. An early survey of the work of these authors is [11], and more recent treatments are [2] and [17]. For a lively historical account of the development of the subject, we refer the reader to [25], in which it is noted that elements of the theory, including in particular equation (2.13), can be traced back at least as far the work of Drach [9] in 1919. Here we follow the approach of Gel'fand and Dickey, which first appeared in [16], and has received a nice expository treatment in Chapter 12 of Dickey's book [7]. In this approach, the stationary equations, which have the structure of completely integrable Hamiltonian systems, are rewritten in action-angle variables, which reduces them to an easily integrable set of equations (see (5.22) below) first obtained by Dubrovin in [10]. (We remark that each stationary equation is a finite-dimensional completely integrable Hamiltonian system in the classical sense; unlike the time-dependent KdV equations which are in some sense [13, 14] infinite-dimensional completely integrable Hamiltonian systems.) Integrating Dubrovin's equations shows that every smooth solution of the stationary equations must be expressible in the form given below in (3.22), which is known as the Its-Matveev formula [20]. It turns out that this part of the proof is valid for all stationary equations for the KdV hierarchy. We then conclude by determining which solutions of the Its-Matveev formula are nonsingular. The latter step we have so far only completed for the second stationary equation in the hierarchy: that is, for the equation for 2-solitons.

We emphasize that our interest here is not in constructing solutions of the stationary equations; all the solutions appearing in this paper are already well-known (see, for example, [26]). Rather, our focus is on showing that a corollary of the method used to integrate these equations is that the N-solution solutions are the *only* solutions with finite energy, at least in the case $N = 2$. Also, we have made an effort to give a self-contained presentation, which in fact relies entirely on elementary calculations.

The result we prove here has consequences for the stability theory of KdV multisolitons. As we show in a forthcoming paper, it can be used to show that

two-soliton solutions of KdV are global minimizers for the third invariant of the KdV equation, subject to the constraint that the first two invariants be held constant. This in turn establishes the stability of two-soliton solutions, thus providing an alternative proof to that appearing in [28].

We remark that in order to be useful for the stability theory, it is important that our uniqueness result make no assumption on the values of the parameters d_i appearing in equation (5.1). This requirement influenced our choice of method of proof. An alternate method we considered was to proceed by an argument which counts the dimensions of the stable and unstable manifolds of (5.1) at the origin in phase space. Indeed, if one assumes in advance that d_3 and d_5 are such that equation (5.5) has distinct positive roots, then this method does give, after some work, that the well-known 2-soliton solutions are the only homoclinic solutions of (5.1). Such an argument, however, becomes much more complicated for other choices of d_i, partly because center manifolds of dimension up to 4 can appear. For this reason, we have found it better to proceed by direct integration of the equation instead.

The plan of the paper is as follows. In Sections 2 and 3, for the reader's convenience and to set notation, we review some of the basic properties of multisoliton solutions. In Section 2 we introduce the equations of the KdV hierarchy, and their associated stationary equations. (Here "stationary" means "time-independent": stationary equations are equations for time-independent solutions of the KdV hierarchy. Coincidentally, they are also equations for stationary points of variational problems.) In Section 3 we define the N-soliton solutions of the KdV hierarchy, and give a proof of the well-known fact that their profiles are actually solutions of stationary equations. Section 4 prepares for the main result by treating the elementary case of stationary solutions of the KdV equation itself. In Section 5 we prove the main result, which is that for the stationary equation for the fifth-order equation in the KdV hierarchy, the only H^2 solutions are 1-soliton and 2-soliton profiles. A concluding section discusses the question of how to generalize the result to higher equations in the hierarchy and N-solitons for $N > 2$.

2   The KdV hierarchy

We review here the definition of the KdV hierarchy, following the treatment of chapter 1 of [7].

Let $\mathcal{A}$ denote the differential algebra over $\mathbf{C}$ of formal polynomials in u and the derivatives of u. That is, elements of $\mathcal{A}$ are polynomials with complex coefficients in the symbols u, u', u'', etc.; and elements of $\mathcal{A}$ can be acted on by a derivation ξ, a linear operator on $\mathcal{A}$ which obeys the Leibniz product rule, and takes u to u', u' to u'', etc. We adopt the convention that primes also denote the action of ξ on any element of $\mathcal{A}$. Thus the expressions a' and ξa are synonymous for $a \in \mathcal{A}$. Later we will substitute actual functions of x for u, and then ξ will correspond to the operation of differentiation with respect to x, so that u', u'', etc., will denote the derivatives of these functions with respect to x

in the usual sense.

If M is an integer, we define a pseudo-differential operator of order M to be a formal sum

$$X = \sum_{i=-\infty}^{M} a_i \partial^i, \tag{2.1}$$

where $a_i \in \mathcal{A}$ for each i. Clearly the set $\mathcal{P}$ of all pseudo-differential operators has a natural module structure over the ring $\mathcal{A}$. We can also make $\mathcal{P}$ into an algebra by first defining, for each integer k and each $a \in \mathcal{A}$, the product $\partial^k a$ as

$$\partial^k a = a \partial^k + \binom{k}{1} a' \partial^{k-1} + \binom{k}{2} a'' \partial^{k-2} + \dots,$$

where

$$\binom{k}{i} = \frac{k(k-1)\cdots(k-i+1)}{i!};$$

and then extending this multiplication operation to all of $\mathcal{P}$ in the natural way:

$$\left(\sum_{i=-\infty}^{M} a_i \partial^i \right) \left(\sum_{j=-\infty}^{N} b_j \partial^j \right) = \sum_{i=-\infty}^{M} \sum_{j=-\infty}^{N} a_i (\partial^i b_j) \partial^j$$

$$= \sum_{i=-\infty}^{M} \sum_{j=-\infty}^{N} \sum_{l=0}^{\infty} a_i \binom{i}{l} (\xi^l b_j) \partial^{i+j-l}.$$

The last sum in the preceding equation is well-defined in $\mathcal{P}$ because each value of $i+j-l$ occurs for only finitely many values of the indices i, j, l. It can be checked that, with this definition of multiplication, $\mathcal{P}$ is an associative algebra with derivation ∂. Interestingly, this algebra $\mathcal{P}$ was studied by Schur in [31], many years before its utility for the theory of integrable systems was discovered.

In particular we will have $\partial \partial^{-1} = 1$ in $\mathcal{P}$. More generally, suppose X is given by (2.1) with $a_M = 1$. Then X has a multiplicative inverse X^{-1} in $\mathcal{P}$; this may be verified by first observing that the order of X^{-1} must be $-M$ and then using the equation $XX^{-1} = 1$ to solve recursively for the coefficients b_i of $X^{-1} = \sum_{i=-\infty}^{-M} b_i \partial^i$. Carrying out this process, one finds that the b_i are polynomials in the a_i and their derivatives. Similarly, there exists $Y \in \mathcal{P}$ such that $Y^m = X$, as may be proved by observing that Y must be of order 1, and using the equation $Y^m = X$ to solve recursively for the coefficients c_i of $Y = \sum_{i=-\infty}^{1} c_i \partial^i$. These coefficients will be uniquely determined if we specify that $c_1 = 1$, and in that case the operator Y so obtained will be denoted by $X^{1/m}$. (All other solutions of $Y^m = 1$ are of the form $\alpha X^{1/m}$ where α is an mth root of unity.) For $k \in \mathbf{Z}$ we then define $X^{k/m}$ to be the kth power of $X^{1/m}$. Since X is an integer power of $X^{1/m}$ it follows immediately that X and $X^{1/m}$ commute, and hence so do X and $X^{k/m}$.

If in (2.1) we have $a_i = 0$ for all $i < 0$, then we say that X is a differential operator; obviously the product and sum of any two differential operators is again a differential operator. For general $X \in \mathcal{P}$, the differential part of X,

denoted by X_+, is defined to be the differential operator obtained by omitting all the terms from X which contain ∂^i with negative i. We also define X_- to be $X - X_+$. As usual, we define the commutator $[X_1, X_2]$ of two elements of $\mathcal{P}$ by $[X_1, X_2] = X_1 X_2 - X_2 X_1$. Also, if X is given by (2.1), it will be useful to define the residue of X, $\mathrm{Res}\, X$, to equal a_{-1}. That is, $\mathrm{Res}\, X \in \mathcal{A}$ is the coefficient of ∂^{-1} in the expansion of X. Finally, for X as in (2.1) we define $\sigma_2(X) \in \mathcal{P}$ by

$$\sigma_2(X) = a_{-1}\, \partial^{-1} + a_{-2}\, \partial^{-2}. \tag{2.2}$$

The Korteweg-de Vries hierarchy can be defined in terms of fractional powers of the differential operator L given by

$$L = \partial^2 + u. \tag{2.3}$$

From the above considerations, $L^{(2k+1)/2}$ is well-defined as an element of $\mathcal{P}$ for each nonnegative integer k. When we take its differential part, we obtain the operator $\left(L^{(2k+1)/2}\right)_+$, which has the following important property.

Lemma 2.1. *The commutator $[\left(L^{(2k+1)/2}\right)_+, L]$ is a differential operator of order 0; that is, a polynomial in u and its derivatives. In fact, it is given by the equation*

$$\left[\left(L^{(2k+1)/2}\right)_+, L\right] = 2\left(\mathrm{Res}\, L^{(2k+1)/2}\right)'. \tag{2.4}$$

Proof. As the commutator of two differential operators, $[\left(L^{(2k+1)/2}\right)_+, L]$ is a differential operator. Now

$$[\left(L^{(2k+1)/2}\right)_+, L] = [L^{(2k+1)/2}, L] - [\left(L^{(2k+1)/2}\right)_-, L],$$

and, as noted above, $L^{(2k+1)/2}$ commutes with L, so

$$[\left(L^{(2k+1)/2}\right)_+, L] = -[\left(L^{(2k+1)/2}\right)_-, L]. \tag{2.5}$$

Observe that, in general, the commutator of an operator of order M_1 and an operator of order M_2 has order $M_1 + M_2 - 1$. Since the right hand side of (2.5) is a commutator of an operator of order -1 and an operator of order 2, it therefore has order 0.

Once it is established that both sides of (2.5) are equal to a differential operator of order 0, the identity (2.4) is easily obtained by computing the term of order 0 in the expansion of $-[\left(L^{(2k+1)/2}\right)_-, L]$. $\qquad \square$

The Korteweg-de Vries hierarchy is a set of partial differential equations, indexed by the natural numbers $k = 0, 1, 2, 3, \ldots$, for functions $u(x, t_{2k+1})$ of two real variables x and t_{2k+1}. The kth equation in the hierarchy is defined as

$$u_{t_{2k+1}} = 2\left(\mathrm{Res}\, L^{(2k+1)/2}\right)'. \tag{2.6}$$

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Here the subscripted t_{2k+1} denotes the derivative with respect to t_{2k+1}. Starting with $k = 0$, the first three in the hierarchy are given by:

$$u_{t_1} = u',$$
$$u_{t_3} = \frac{1}{4}(u''' + 6uu'),$$
$$u_{t_5} = \frac{1}{16}(u''''' + 10uu''' + 20u'u'' + 30u^2u').$$
(2.7)

The second equation in (2.7) is the KdV equation (1.1).

This definition of the hierarchy is due to Gelfand and Dickey, and leads to simple formulations and proofs of many properties of these equations, including the fact that they define commuting flows, which were formerly proved by more unwieldy methods. Also, natural modifications of the definition lead readily to more general hierarchies of equations (today called Gelfand-Dickey hierarchies), of which the Korteweg-de Vries hierarchy is just one, and which share many of the interesting integrability properties of the Korteweg-de Vries hierarchy [7, 8].

An important feature of the KdV hierarchy (2.6) is that the differential polynomials which appear on the right-hand side satisfy a simple recurrence relation. Following the notation of Chapter 12 of [7], let us define, for $k = 0, 1, 2, \ldots$,

$$R_{2k+1} = \frac{(-1)^k}{2} \operatorname{Res} L^{(2k-1)/2},$$

so that the KdV hierarchy takes the form

$$u_{t_{2k+1}} = 4(-1)^{k+1} R'_{2k+3}.$$
(2.8)

Lemma 2.2. *The differential polynomials R_{2k+1} satisfy the recurrence relation*

$$R'''_{2k+1} + 4uR'_{2k+1} + 2u'R_{2k+1} = -4R'_{2k+3},$$
(2.9)

for $k = 0, 1, 2, \ldots$, with initial condition $R_1 = 1/2$.

Proof. The proof of this lemma is essentially an exercise on the material in Section 1.7 of [7], but for the reader's convenience we indicate the details here.

Let $\mathcal{C}$ denote the set of all formal Laurent series in z of the form $\sum\limits_{r=-\infty}^{\infty} X_r z^r$, where $X_r \in \mathcal{P}$ for $r \in \mathbf{Z}$. Then $\mathcal{C}$ inherits an operation of addition from $\mathcal{P}$, and if S and T are in $\mathcal{C}$ and all but finitely many of the coefficients in T are zero, then the products ST and TS are defined in $\mathcal{C}$ by the usual term-by-term multiplication of series. Also, for $S = \sum_{r=-\infty}^{\infty} X_r z^r$ in $\mathcal{C}$ we define $S_+ = \sum_{r=-\infty}^{\infty} (X_r)_+ z^r$, $S_- = \sum_{r=-\infty}^{\infty} (X_r)_- z^r$, $\operatorname{Res}(S) = \sum_{r=-\infty}^{\infty} (\operatorname{Res} X_r) z^r$, and $\sigma_2(S) = \sum_{r=-\infty}^{\infty} \sigma(X_r) z^r$, where σ_2 is the operator defined in (2.2).

Let $\widehat{L} = L - z^2$, and define the map $H : \mathcal{C} \to \mathcal{C}$ by

$$H(X) = (\widehat{L}X)_+ \widehat{L} - \widehat{L}(X\widehat{L})_+.$$

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(Dickey [7] calls H the Adler map, as it was introduced in section 4 of [1].) Since $(\widehat{LX})\widehat{L} = \widehat{L}(X\widehat{L})$, it follows that $H(X) = (\widehat{LX})_-\widehat{L} - \widehat{L}(X\widehat{L})_-$ for all X in $\mathcal{C}$. Moreover, from the definition of H and the fact that L is a differential operator of order 2, one sees easily that $H(X) = H(\sigma_2(X))$ for all X in $\mathcal{C}$.

Define
$$T = \sum_{r=-\infty}^{\infty} L^{r/2} z^{-r-4}.$$

Clearly $\widehat{L}T = T\widehat{L} = 0$, so $H(T) = 0$, and hence also $H(\sigma_2(T)) = 0$. On the other hand, by observing that $\sigma_2(L^{2k}) = 0$ for all nonnegative integers k, $\sigma_2(L^{r/2}) = 0$ for all integers $r \leq -3$, and $\sigma_2(L^{-1}) = \partial^{-2}$, we can write $\sigma_2(T)$ as

$$\sigma_2(T) = R\partial^{-1} + \tilde{R}\partial^{-2}, \tag{2.10}$$

where R and $\tilde{R}$ are in $\mathcal{C}$ and

$$R = \operatorname{Res} \sigma_2(T) = \sum_{k=0}^{\infty} 2(-1)^k R_{2k+1} \; z^{-2k-3}. \tag{2.11}$$

Substituting (2.10) into the equation $H(\sigma_2(T)) = 0$, we find after a computation that

$$0 = H(\sigma_2(T)) = (-\tilde{R}'' + 2uR' + u'R - 2z^2 R') - (R'' + 2\tilde{R}')\partial, \tag{2.12}$$

and therefore $\tilde{R} = -\frac{1}{2}R'$ and

$$\frac{1}{2}R''' + 2uR' + u'R = 2z^2 R'. \tag{2.13}$$

Then substituting (2.11) into (2.13) gives (2.9) for $k = 0, 1, 2, \ldots$. Finally, we can verify that $R_1 = 1/2$ by directly computing $R_1 = \frac{1}{2} \operatorname{Res} L^{-1/2}$.
$\square$

Using the recurrence relation in Lemma 2.2, we find, for example, that the first few terms in the sequence $\{R_{2k+1}\}$ are

$$
\begin{aligned}
R_1 &= 1/2, \\
R_3 &= (-1/4)u, \\
R_5 &= (1/16)(u'' + 3u^2), \\
R_7 &= (-1/64)(u'''' + 5u'^2 + 10uu'' + 10u^3).
\end{aligned}
\tag{2.14}
$$

Of particular interest are time-independent or stationary solutions of (2.8). If u is a such a solution, then u satisfies (2.8) with $u_t = 0$, and hence integration gives that u satisfies the equation $R_{2k+3} = d$, where d is a constant, independent of x and t. Letting $d_1 = 2d$, we can rewrite this equation in the form

$$d_1 R_1 - R_{2k+3} = 0.$$

More generally, we can view any solution of the equation

$$d_1 R_1 + d_3 R_3 + d_5 R_5 + \cdots + d_{2N+3} R_{2N+3} = 0 \qquad (2.15)$$

as a stationary solution of the equation

$$u_t = d_3 R'_3 + d_5 R'_5 + \cdots + d_{2N+3} R'_{2N+3},$$

which itself can be considered to be an equation in the KdV hierarchy. For this reason, following [7], we refer to equations (2.15) as the stationary equations of the KdV hierarchy. (They are also sometimes called Lax-Novikov equations.)

Equation (2.15) is an ordinary differential equation of order $2N$, and can therefore be rewritten as a first-order system in phase space $\mathbf{R}^{2N}$. It turns out that this system is of Hamiltonian form, and in fact is completely integrable in the sense that it has N independent integrals in involution with each other. In general, Liouville's method provides a technique for actually integrating completely integrable systems: that is, for explicitly finding the transformation from coordinates of phase space to action-angle variables. However, this integration involves solving a system of first-order partial differential equations. For the system (2.15), Dubrovin [10] introduced a change of variables under which this system of PDE's has a simple form and is trivially solvable. This is the change of variables we use below in Section 5.

3   N-soliton profiles

Another key aspect of the KdV hierarchy is that the flows which it defines all commute with each other, at least formally. More precisely, one can check that the equations in (2.8) have the formal structure of Hamiltonian equations with respect to a certain symplectic form, and are all in involution with each other with respect to this form (see, for example, chapters 1 through 4 of [7]). This suggests the following. Assume that a function class $\mathcal{S}$ has been defined such that for each $k \in \mathbf{N}$, the initial-value problem for equation (2.8) is well-posed on $\mathcal{S}$, and let $S(t_{2k+1})$ be the solution map for this problem, which to each $\psi \in \mathcal{S}$ assigns the function $S(t_{2k+1})[\psi] = u(\cdot, t_{2k+1}) \in \mathcal{S}$, where u is the solution of (2.8) with initial data $u(x, 0) = \psi(x)$. Then in light of the formal structure mentioned above, one would expect that the solution operators $S(t_{2k+1})$ and $S(t_{2l+1})$ commute with each other as mappings on $\mathcal{S}$. Hence, for each $\psi \in \mathcal{S}$ and each $l \in \mathbf{N}$, one should be able to define a simultaneous solution $u(x, t_1, t_3, t_5, \ldots, t_{2l+1})$ to all of the first l equations in the hierarchy by setting

$$u(x, t_1, t_3, \ldots, t_{2l+1}) = S(t_1) S(t_3) \cdots S(t_{2l+1}) \psi.$$

This formal analysis, however, does not lead easily to concrete results about general solutions of the KdV hierarchy. For this reason there has historically been great interest in constructing and elucidating the structure of explicit solutions. In this section we review the definition and basic properties of an important class of such solutions, the N-soliton solutions.

To begin the construction of N-soliton solutions, let $N \in \mathbf{N}$, and for $1 \leq j \leq N$ define the functions

$$y_j(x) = e^{\alpha_j x} + a_j e^{-\alpha_j x}, \tag{3.1}$$

where α_j and a_j are complex numbers satisfying

    (i) for all $j \in \{1, \ldots, N\}$, $\alpha_j \neq 0$ and $a_j \neq 0$,
    (ii) for all $j, k \in \{1, \ldots, N\}$, if $j < k$ then $\alpha_j \neq \alpha_k$ and $0 \leq \mathfrak{Re}\,\alpha_j \leq \mathfrak{Re}\,\alpha_k$.
$$\tag{3.2}$$

We will use $D(y_1, \ldots, y_N)$ to denote the Wronskian of $y_1, \ldots y_N$:

$$D(y_1, \ldots, y_N) = \begin{vmatrix} y_1 & \cdots & y_N \\ y_1' & \cdots & y_N' \\ \cdots & \cdots & \cdots \\ y_1^{(N-1)} & \cdots & y_N^{(N-1)} \end{vmatrix}. \tag{3.3}$$

Next we will construct an operator of the form (2.3) from the y_j, using a technique known as the "dressing method" [7]. First, on any interval I where $D \neq 0$, we define a differential operator ϕ of order N by

$$\phi = \frac{1}{D} \begin{vmatrix} y_1 & \cdots & y_N & 1 \\ y_1' & \cdots & y_N' & \partial \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(N-1)} & \cdots & y_N^{(N-1)} & \partial^{N-1} \\ y_1^{(N)} & \cdots & y_N^{(N)} & \partial^N \end{vmatrix}. \tag{3.4}$$

Here it is understood that the determinant in (3.4) is to be expanded along the final column, multiplying each operator ∂^i by its corresponding cofactor on the left. In other words,

$$\phi = \partial^N + W_{N-1}\partial^{N-1} + W_{N-2}\partial^{N-2} + \cdots + W_1\partial + W_0, \tag{3.5}$$

where for $i = 1, \ldots, N-1$,

$$W_i = \frac{(-1)^{N+2+i}}{D} \begin{vmatrix} y_1 & \cdots & y_N \\ y_1' & \cdots & y_N' \\ \cdots & \cdots & \cdots \\ y_1^{(i-1)} & \cdots & y_N^{(i-1)} \\ y_1^{(i+1)} & \cdots & y_N^{(i+1)} \\ \cdots & \cdots & \cdots \\ y_1^{(N)} & \cdots & y_N^{(N)} \end{vmatrix}, \tag{3.6}$$

and

$$W_0 = \frac{(-1)^{N+2}}{D} \begin{vmatrix} y_1' & \cdots & y_N' \\ y_1'' & \cdots & y_N'' \\ \cdots & \cdots & \cdots \\ y_1^{(N)} & \cdots & y_N^{(N)} \end{vmatrix}. \tag{3.7}$$

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Next, we slightly generalize the notion of pseudo-differential operator defined in Section 2 to include formal sums of type (2.1) in which the a_i are no longer differential polynomials in a single variable u, but now are rational functions of the n symbols $y_1, y_2, \ldots, y_n$ and their formal derivatives y_1', y_1'', y_2', y_2'', etc. (forgetting for the moment that $y_1, y_2, \ldots, y_n$ are actually functions of x). The definitions of the multiplication and inverse operations on pseudo-differential operators given in Section 2 remain unchanged for this larger algebra. Thus, ϕ as defined in (3.4) has a formal inverse ϕ^{-1}, which is a pseudo-differential operator whose coefficients are rational functions of y_i and their derivatives, expressible as polynomials in W_j and their derivatives. We now define L as the formal pseudo-differential operator given by

$$L = \phi \partial^2 \phi^{-1}. \tag{3.8}$$

Lemma 3.1. *The differential part of L is*

$$L_+ = \partial^2 - 2W_{N-1}'. \tag{3.9}$$

Proof. First observe that

$$L = (\phi \partial^{-N}) \partial^2 (\phi \partial^{-N})^{-1}. \tag{3.10}$$

Now we can write

$$\phi \partial^{-N} = 1 + W_{N-1} \partial^{-1} + W_{N-2} \partial^{-2} + O(\partial^{-3}), \tag{3.11}$$

where "$O(\partial^{-3})$" denotes terms containing ∂^j with $j \leq -3$. Also, a computation shows that

$$(\phi \partial^{-N})^{-1} = 1 - W_{N-1} \partial^{-1} + \left(W_{N-1}^2 - W_{N-2} \right) \partial^{-2} + O(\partial^{-3}). \tag{3.12}$$

Equation (3.9) then follows easily by inserting (3.11) and (3.12) into (3.10) and carrying out the multiplication to determine the terms of nonnegative order. $\square$

Lemma 3.2. *Define $W_{-1} = 0$. Then*

$$L_- \phi = -\sum_{j=0}^{N-1} \left(W_j'' + 2W_{j-1}' - 2W_{N-1}' W_j \right) \partial^j. \tag{3.13}$$

Proof. Since $L_- = L - L_+$, we have from (3.8) that

$$L_- \phi = \phi \partial^2 - L_+ \phi. \tag{3.14}$$

The desired result follows by substituting (3.5) and (3.9) into the right-hand side, and carrying out the multiplications. $\square$

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So far, in discussing ϕ and L, we have considered them only as formal pseudo-differential operators with coefficients that are rational functions in the symbols y_i, y_i', y_i'',.... Now, however, we wish to "remember" the fact that these coefficients are specific functions of x. To this end we first observe that by (3.2) the functions $y_1, \ldots, y_N$ are analytic and linearly independent. Therefore, by a theorem of Peano (see [3]), their Wronskian D cannot vanish identically on any open interval in $\mathbf{R}$. In particular, by continuity there exists an open interval I on $\mathbf{R}$ such that $D(x) \neq 0$ for all $x \in I$. Therefore, on I the right-hand-side of (3.4) defines a linear differential operator with smooth coefficients W_i for $i = 1, \ldots, N-1$. To emphasize the distinction between the formal operator ϕ and its concrete realization, we introduce the notation $r(\phi)$ for the differential operator with smooth coefficients on I obtained by remembering that the y_i are certain functions of x.

More generally, if X is any pseudo-differential operator whose coefficients are formal polynomials in W_i and their derivatives, we define $r(X)$ to be the operator obtained by remembering that the coefficients of X are actually smooth functions of x on I. Thus r defines an algebra homomorphism from $\mathcal{P}$ to the to the algebra of pseudo-differential operators with coefficients that are smooth functions on I.

Although it is clear from Lemma 3.2 that L_- is not zero as a formal pseudo-differential operator, nevertheless the coefficients of L_- evaluate to zero when viewed as functions on I. That is, we have the following result.

Lemma 3.3. *When L is defined as in (3.8), with ϕ given by (3.4), then*

$$r(L_-) = 0.$$

Proof. From (3.13) we have that, as a formal pseudo-differential operator,

$$L_-\phi = \sum_{j=0}^{N-1} F_j \partial^j,$$

where each F_j is a differential polynomial in $W_0, \ldots, W_{N-1}$. Therefore

$$r(L_-\phi) = \sum_{j=0}^{N-1} F_j(x)\partial^j,$$

where the $F_j(x)$ are smooth functions on I.

For all $i = 1, \ldots, N$, we see from (3.4) that

$$r(\phi)y_i = 0, \tag{3.15}$$

and since $\partial^2 y_i = \alpha_i^2 y_i$, then $r(\phi\partial^2)y_i = 0$ also. Therefore (3.14) implies that

$$r(L_-\phi)y_i = \sum_{j=0}^{N-1} F_j(x)\partial_j y_i(x) = 0 \text{ for } i = 1, \ldots, N. \tag{3.16}$$

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Since $D(x) = \det(\partial_j y_i(x)) \neq 0$ for all $x \in I$, it follows from (3.16) that $F_j(x) = 0$ for all $x \in I$ and all $j = 0, 1, \ldots, N-1$.

Now as a formal pseudo-differential operator, ϕ is invertible, with inverse ϕ^{-1} of the form

$$\phi^{-1} = \sum_{k=0}^{\infty} B_k \partial^{-N-k},$$

where the B_k are differential polynomials in $W_0, \ldots, W_{N-1}$. Hence

$$L_- = (L_-\phi)\phi^{-1} = \sum_{j=0}^{N-1} F_j \partial^j \sum_{k=0}^{\infty} B_k \partial^{-N-k} = \sum_{r=1}^{\infty} \sum_{j=0}^{N-1} (F_j G_{j,r}) \partial^{N-r},$$

where each $G_{j,r}$ is a finite linear combination of the B_k and their formal derivatives B_k', B_k'', ... Since $F_j(x) = 0$ for all $x \in I$, it follows that the coefficients $\sum_{j=0}^{N-1} (F_j(x) G_{j,r}(x))$ of $r(L_-)$ are also identically zero on I. $\qquad\square$

The following consequence of Lemma 3.3 will be useful in Section 5.

Corollary 3.4. *If I is any interval such that $D(x) \neq 0$ for all $x \in I$, then the equation*

$$W_{N-1}' - W_{N-1}^2 + 2W_{N-2} + \sum_{i=1}^{N} \alpha_i^2 = 0 \tag{3.17}$$

holds at all points of I.

Proof. From Lemma 3.3 we have $r(L_-) = 0$, and hence each coefficient in the sum in (3.13) is identically zero as a function of x on I. In particular,

$$W_{N-1}'' + 2W_{N-2}' - 2W_{N-1}'W_{N-1} = 0$$

on I. Integrating gives

$$W_{N-1}' + 2W_{N-2} - W_{N-1}^2 + C = 0,$$

where C is a constant.

To evaluate C, first assume that α_1, ..., α_N are positive numbers, and observe that since y_i behaves as $x \to \infty$ like $e^{\alpha_i x}$, we have that $\lim_{x \to \infty} W_{N-1} = -d_1/d$, $\lim_{x \to \infty} W_{N-1}' = 0$, and $\lim_{x \to \infty} W_{N-2} = d_2/d$, where

$$d = \begin{vmatrix} 1 & \ldots & 1 \\ \alpha_1 & \ldots & \alpha_N \\ \alpha_1^2 & \ldots & \alpha_N^2 \\ \ldots & \ldots & \ldots \\ \alpha_1^{N-1} & \ldots & \alpha_N^{N-1} \end{vmatrix}$$

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is the Vandermonde matrix of the numbers $\alpha_1, \ldots, \alpha_N$, and

$$
d_1 = \begin{vmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_N \\ \alpha_1^2 & \cdots & \alpha_N^2 \\ \cdots & \cdots & \cdots \\ \alpha_1^{N-2} & \cdots & \alpha_N^{N-2} \\ \alpha_1^{N} & \cdots & \alpha_N^{N} \end{vmatrix}, \quad
d_2 = \begin{vmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_N \\ \alpha_1^2 & \cdots & \alpha_N^2 \\ \cdots & \cdots & \cdots \\ \alpha_1^{N-3} & \cdots & \alpha_N^{N-3} \\ \alpha_1^{N-1} & \cdots & \alpha_N^{N-1} \\ \alpha_1^{N} & \cdots & \alpha_N^{N} \end{vmatrix}.
$$

Therefore

$$
C = \lim_{x \to \infty} \left(W_{N-1}^2 - 2W_{N-2} - W_{N-1}' \right) = \left(\frac{d_1}{d} \right)^2 - 2 \left(\frac{d_2}{d} \right). \tag{3.18}
$$

But it follows from a classic exercise on Vandermonde matrices (see problem 10 on p. 99 of [30], or [27]) that

$$
d_1/d = \sum_{i=1}^{n} \alpha_i \quad \text{and} \quad d_2/d = \sum_{1 \le i < j \le n} \alpha_i \alpha_j.
$$

Substituting in (3.18), we obtain $C = \sum_{i=1}^{N} \alpha_i^2$, as desired. The result for general complex values of $\alpha_1, \ldots, \alpha_N$ then follows by analytic continuation. $\square$

Remark. Since (3.17) is a Ricatti equation, the substitution $W_{N-1} = -D'/D$ converts it to the following linear equation for D:

$$
D'' = \left(2W_{N-2} + \sum_{i=1}^{N} \alpha_i^2 \right) D. \tag{3.19}
$$

Corollary 3.4 is therefore equivalent to the assertion that (3.19) holds when D is given by (3.3) and (3.1).

Corollary 3.5. *Suppose D is given by (3.3), ϕ by (3.4), and L by (3.8). Then*

$$
r(L) = \partial^2 + u,
$$

where

$$
u = 2 \left(\frac{D'}{D} \right)'. \tag{3.20}
$$

Proof. From the definition of the determinant and the product rule, one easily sees that the derivative of D is given by

$$
D' = \begin{vmatrix} y_1 & \cdots & y_N \\ y_1' & \cdots & y_N' \\ \cdots & \cdots & \cdots \\ y_1^{(N-2)} & \cdots & y_N^{(N-2)} \\ y_1^{(N)} & \cdots & y_N^{(N)} \end{vmatrix}, \tag{3.21}
$$

which, as we see from (3.6), implies that $W_{N-1} = -D'/D$. Since $r(L) = r(L_+) + r(L_-)$, the desired result therefore follows from Lemma 3.1 and Lemma 3.3. $\square$

Definition 3.6. *Let y_j be given by (3.1), and assume (3.2) holds. Then we define*

$$\psi^{(N)}(x) = \psi^{(N)}(x; a_1, \ldots, a_N; \alpha_1, \ldots, \alpha_N) = 2 \left(\frac{D'(y_1, \ldots, y_N)}{D(y_1, \ldots, y_N)} \right)' \quad (3.22)$$

for all x such that $D(y_1, \ldots, y_N) \neq 0$.

Introducing simple time dependencies into $\psi^{(N)}$ yields a function which satisfies all the equations in the KdV hierarchy simultaneously.

Theorem 3.7. *Let $N \in \mathbf{N}$, and let a_j, and α_j, $j = 1, \ldots, N$ be complex numbers satisfying (3.2). Fix $l \in \mathbf{N}$, and for $1 \leq j \leq N$, define the function $\tilde{a}_j$ by*

$$\tilde{a}_j(x, t_3, t_5, t_7, \ldots, t_{2l+1}) = a_j \exp\left(-2(\alpha_j^3 t_3 + \alpha_j^5 t_5 + \cdots + \alpha_j^{2l+1} t_{2l+1}) \right). \quad (3.23)$$

Then let u be defined as a function of x, t_3, t_5, $\ldots$, t_{2l+1} by

$$u = \psi^{(N)}(x; \tilde{a}_1, \ldots, \tilde{a}_N; \alpha_1, \ldots, \alpha_N). \quad (3.24)$$

Then for all $k \in \{1, \ldots, l\}$, and at all points in its domain of definition, u satisfies the partial differential equation

$$\frac{\partial u}{\partial t_{2k+1}} = \left[\left(L^{(2k+1)/2} \right)_+, L \right] = 4(-1)^{k+1} R'_{2k+3}. \quad (3.25)$$

Remark. Using the fact that multiplication of y_i by the exponential of a linear function of x does not change the value of $\psi^{(N)}$, one sees easily that (3.24) can also be written in the form

$$u = 2 \left(\frac{D'(\tilde{y}_1, \ldots, \tilde{y}_N)}{D(\tilde{y}_1, \ldots, \tilde{y}_N)} \right)',$$

where $\tilde{y}_j(x) = e^{w_j} + a_j e^{-w_j}$ and $w_j = \alpha_j x + \alpha_j^3 t_3 + \alpha_j^5 t_5 + \cdots + \alpha_j^{2l+1} t_{2l+1}$.

We will not need to make use of Theorem 3.7 in the present paper, and so do not include a proof here. But the reader may be interested to know that, using the tools defined in Section 2 above, a one-paragraph proof can be given. It may be found in [7], where it appears as the proof of part (ii) of Proposition 1.6.5 in [7], or in [8] as the proof of Proposition 1.7.5.

We are concerned here, rather, with the fact the functions $\psi^{(N)}$ satisfy stationary equations of the form (2.15):

Theorem 3.8. *Let $\psi^{(N)}$ be as in Definition 3.6, and define constants s_0, s_1, ..., s_N by*

$$s_0 + s_1 x + s_2 x^2 + \cdots + s_N x^N = (x - \alpha_1^2)(x - \alpha_2^2) \cdots (x - \alpha_N^2); \qquad (3.26)$$

in other words, s_i is the ith elementary symmetric function of N variables, evaluated on $-\alpha_1^2$, ..., $-\alpha_N^2$. Then, on each interval of its domain of definition, the function

$$u(x) = \psi^{(N)}(x; a_1, \ldots, a_N; \alpha_1, \ldots, \alpha_N)$$

satisfies the ordinary differential equation in x given by

$$s_0 R_3 - s_1 R_5 + s_2 R_7 + \cdots + (-1)^N s_N R_{2N+3} = C, \qquad (3.27)$$

where C is a constant.

Proof. Define $\tilde{a}_i$ by (3.23) with $l = N$, and for $i = 1, \ldots, N$ extend y_i to be a function of $x, t_3, t_5, \ldots, t_{2N+1}$ by replacing a_i with $\tilde{a}_i$ in (3.1). That is, set

$$y_i = \exp(\alpha_i x) + a_i \exp\left[-(\alpha_i x + 2\alpha_i^3 t_3 + 2\alpha_i^5 t_5 + \cdots + 2\alpha_i^{2N+1} t_{2N+1})\right]. \quad (3.28)$$

Also extend u to be a function of $x, t_3, t_5, \ldots, t_{2N+1}$ by (3.24). With ϕ and W_i defined in terms of y_i as before, we have as in (3.20) that

$$u = -2W'_{N-1}. \qquad (3.29)$$

For $1 \le k \le N$, let ∂_{2k+1} denote differentiation with respect to t_{2k+1}.

For each i from 1 to N, we apply to both sides of (3.15) the operator

$$\tilde{\partial} = s_0 \partial + s_1 \partial_3 + s_2 \partial_5 + \cdots + s_N \partial_{2N+1}.$$

There results the identity

$$0 = (\tilde{\partial} r(\phi)) y_i + r(\phi)(\tilde{\partial} y_i), \qquad (3.30)$$

where

$$\tilde{\partial} r(\phi) = (\tilde{\partial} W_{N-1}) \partial^{N-1} + (\tilde{\partial} W_{N-2}) \partial^{N-2} + \cdots + (\tilde{\partial} W_1) \partial + \tilde{\partial} W_0.$$

But for all $k = 1, \ldots, N$ we have from (3.28) that

$$\partial_{2k+1} y_i = \alpha_i^{2k+1} z_i,$$

where

$$z_i = -2a_i \exp\left[-(\alpha_i x + 2\alpha_i^3 t_3 + 2\alpha_i^5 t_5 + \cdots + 2\alpha_i^{2N+1} t_{2N+1})\right].$$

Therefore

$$\tilde{\partial} y_i = \left(s_0 \alpha_i + s_1 \alpha_i^3 + s_2 \alpha_i^5 + \cdots + s_N \alpha_i^{2N+1}\right) z_i. \qquad (3.31)$$

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It follows from (3.26) and (3.31) that $\tilde{\partial} y_i = 0$, and so, by (3.30),

$$(\tilde{\partial} r(\phi)) y_i = 0.$$

Now $\tilde{\partial} r(\phi)$ is a linear differential operator in x of order $N - 1$ or less, so as in the proof of Lemma 3.3, the fact that it takes all the functions $y_1, \ldots, y_N$ to zero means that all its coefficients must be identically zero. In particular, $\tilde{\partial} W_{N-1} = 0$, or, in other words,

$$s_0 \partial W_{N-1} + s_1 \partial_3 W_{N-1} + \cdots + s_N \partial_{2N+1} W_{N-1} = 0. \tag{3.32}$$

On the other hand, (3.25) and (3.29) tell us that

$$-2 \partial_{2k+1} W'_{N-1} = 4(-1)^{k+1} R'_{2k+3} \tag{3.33}$$

for $1 \le k \le l$. Integrating (3.33) with respect to x, we obtain

$$\partial_{2k+1} W_{N-1} = 2(-1)^k R_{2k+3} + C_k, \tag{3.34}$$

for $1 \le k \le l$, where C_k is a constant of integration. Moreover, from (2.9) we have $R_3 = -u/4$, and hence

$$\partial W_{N-1} = W'_{N-1} = 2R_3. \tag{3.35}$$

Substituting (3.34) and (3.35) into (3.32) then gives (3.27). $\qquad \square$

In general, $\psi^{(N)}$ will have singularities at points where the denominator D in (3.4) is equal to zero, but away from these points, $\psi^{(N)}$ is a smooth, and in fact analytic, function of its arguments. Our next task is to determine conditions on the parameters α_i and a_i under which D has no zeros on $\mathbf{R}$, or equivalently under which $\psi^{(N)}$ is a smooth function on all of $\mathbf{R}$.

For this purpose it will be useful to represent D as an explicit sum of exponential functions. For given $N \in \mathbf{N}$, let $\{-1, 1\}^N$ denote the set of functions ϵ from $\{1, \ldots, N\}$ to $\{-1, 1\}$; thus $\{-1, 1\}^N$ has cardinality 2^N. For $\epsilon \in \{-1, 1\}^N$, we denote the image of j under ϵ by ϵ_j, and define $S(\epsilon)$ to be the set of all $j \in \{1, \ldots, N\}$ such that $\epsilon_j = -1$. Also, for any ordered N-tuple $(r_1, \ldots, r_N)$, let

$$V(r_1, \ldots, r_N) = \det\{r_i^{j-1}\}_{i,j=1,N} = \prod_{1 \le i < j \le N} (r_j - r_i)$$

be the corresponding Vandermonde determinant. Then expansion of the determinant in (3.3) yields the formula

$$D = \sum_{\epsilon \in \{-1,1\}^N} \left[\exp\left(\sum_{j=1}^N \epsilon_j \alpha_j x \right) \left(\prod_{j \in S(\epsilon)} a_j \right) V(\epsilon_1 \alpha_1, \ldots, \epsilon_N \alpha_N) \right]. \tag{3.36}$$

Lemma 3.9. *Let y_j be given by (3.1), and suppose that (3.2) holds. Suppose in addition that for all $j \in \{1, \ldots, N\}$, $\mathfrak{Re}\,\alpha_j > 0$. Then $\psi^{(N)}$ and all of its derivatives are defined for all sufficiently large $|x|$, and approach zero exponentially fast as $|x| \to \infty$.*

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Proof. Because $\Re\,\alpha_j > 0$ for all j, the dominant term in (3.36) is Ve^{Sx}, where $V = V(\alpha_1, \ldots, \alpha_N)$ and $S = \sum_{j=1}^{N} \alpha_j$: all other terms have exponents with smaller real parts. In particular, we have $D(x) \neq 0$ whenever $|x|$ is sufficiently large. The dominant terms in $D'(x)$ and $D''(x)$ are VSe^{Sx} and VS^2e^{Sx}, respectively; and so in the expression $\psi^{(N)}(x) = 2(DD'' - (D')^2)/D^2$, the denominator has dominant term V^2e^{2Sx}, while the coefficient of e^{2Sx} in the numerator is zero. It follows easily that $\psi^{(N)}(x)$, together with all its derivatives, tends to zero exponentially fast as $x \to \infty$. A similar argument applies as $x \to -\infty$ (where the dominant term in $D(x)$ is $|a_1 \cdots a_N V| e^{|Sx|}$). $\qquad\square$

Lemma 3.10. *Let y_j be given by (3.1), and suppose that (3.2) holds. Suppose also that for all $j \in \{1, \ldots, N\}$, α_j and a_j are real, and*

$$(-1)^{j-1}a_j > 0. \tag{3.37}$$

Then $\psi^{(N)}(x) \in H^1(\mathbf{R})$.

Proof. Suppose that the α_j and a_j are real and (3.37) holds. Then for $1 \leq j < k \leq N$ the factor $\epsilon_k\alpha_k - \epsilon_j\alpha_j$ in $V(\epsilon_1\alpha_1, \ldots, \epsilon_N\alpha_N)$ has the same sign as ϵ_k. For a given k, there are $k-1$ such factors in $V(\epsilon_1\alpha_1, \ldots, \epsilon_N\alpha_N)$, corresponding to the values $1 \leq j \leq k-1$, so the sign of $V(\epsilon_1\alpha_1, \ldots, \epsilon_N\alpha_N)$ is $\prod_{k \in S(\epsilon)}(-1)^{k-1}$. It then follows from (3.37) that the coefficient of each exponential in (3.36) is positive. Hence $D > 0$ for all $x \in \mathbf{R}$, and it follows that $\psi^{(N)}(x)$ is well-defined and smooth on all of $\mathbf{R}$. Then from Lemma 3.9 it follows that $\psi^{(N)} \in H^1$. $\qquad\square$

Lemma 3.11. *Let y_j be given by (3.1), and suppose that (3.2) holds. Suppose also that for each $j \in \{1, \ldots, N\}$, either (i) α_j and a_j are real, (ii) α_j is purely imaginary and $|a_j| = 1$, or (iii) there exists $k \in \{1, \ldots, N\}$ such that $\alpha_k = \alpha_j^*$ and $a_k = a_j^*$. (These conditions can be summarized by saying that the numbers α_j^2 and $(\log a_j)^2$ are either real and of the same sign, or occur in complex conjugate pairs.) Then $\psi^{(N)}(x)$ is real-valued at all points where it is defined.*

Proof. In case (i) we have $y_j^* = y_j$, in case (ii) we have $y_j^* = (1/a_j)y_j$, and in case (iii) we have $y_j^* = y_k$. It follows easily that the conjugate D^* of $D = D(y_1, \ldots, y_N)$ is equal to a constant times D itself. Therefore $(\psi^{(N)})^* = \psi^{(N)}$. $\qquad\square$

Definition 3.12. *We say that $\psi^{(N)}(x; a_1, \ldots, a_N; \alpha_1, \ldots, \alpha_N)$ is an N-soliton profile if $\psi^{(N)}(x)$ is real-valued for all $x \in \mathbf{R}$, and $\psi^{(N)}(x) \in H^1(\mathbf{R})$. The corresponding time-dependent functions given by (3.24) are called N-soliton solutions of the KdV hierarchy. The numbers $\alpha_1, \ldots, \alpha_N$ are called the wavespeeds of the N-soliton solution.*

Remarks. (i) At least in the case when $N = 2$, it can be shown that, given that the conditions in (3.2) hold for a_j and α_j, the hypotheses on a_j and α_j in Lemmas 3.10 and 3.11 are not only sufficient for $\psi^{(N)}$ to be an N-soliton profile according to the above definition, but also necessary. We conjecture that these

conditions on a_j and α_j are also necessary in the case of general N, although we have not proved this yet.

(ii) If the conjecture in the preceding remark is true, it then follows from Lemma 3.9 that N-soliton profiles, together with all their derivatives, approach zero exponentially fast as $|x| \to \infty$.

(iii) By transforming the index in the outermost sum of (3.36) from ϵ to μ, where $\mu_j = \frac{1}{2}(\epsilon_j + 1)$ for $j \in \{1, \ldots, N\}$, one can rewrite D in the form $D = e^{px+q} D_1$, where p and q are constants and

$$D_1 = \sum_{\mu \in \{0,1\}^N} \exp\left(\sum_{i=1}^{N} 2\mu_i \alpha_i (x + \zeta_i) + \sum_{1 \le i < j \le n} \mu_i \mu_j A_{ij} \right),$$

where ζ_i and A_{ij} are real constants. Explicitly, one has

$$e^{px+q} = \exp\left(-\Sigma_{i=1}^{N} \alpha_i x \right) V(\alpha_1, \ldots, \alpha_N) \Pi_{j=1}^{N} |a_j|,$$

$$\zeta_i = \frac{1}{2\alpha_i} \log \left| \frac{V(\alpha_1, \ldots, \alpha_{i-1}, -\alpha_i, \alpha_{i+1}, \ldots, \alpha_N)}{a_i V(\alpha_1, \ldots, \alpha_N)} \right|, \quad (i = 1, \ldots, N)$$

$$A_{ij} = 2 \log \left| \frac{\alpha_j - \alpha_i}{\alpha_j + \alpha_i} \right|, \quad (i, j = 1, \ldots, N).$$

(Here use has been made of the assumption (3.37).) Writing

$$\psi^{(N)} = 2(D_1'/D_1)',$$

one obtains the formula for N-soliton profile found in [28] or on page 55 of [19].

(iv) If $\psi^{(N)}$ is an N-soliton solution, then the constant C in equation (3.27) is equal to zero; that is,

$$s_0 R_3 - s_1 R_5 + s_2 R_7 + \cdots + (-1)^N s_N R_{2N+3} = 0 \qquad (3.38)$$

on $\mathbf{R}$. This is seen by taking the limit of (3.27) as $x \to \infty$, and observing that, for each $k \ge 1$, R_{2k+1} is a differential polynomial in $u = \psi^{(N)}$ and its derivatives, with no constant term.

4   The stationary equation for $N = 1$

To set the stage for the analysis of (2.15) in the case $N = 2$, we now discuss the case $N = 1$. The result we prove in this section, Theorem 4.2, is a standard exercise in elementary integration, but writing out the proof in detail will serve to introduce the notation we use for the more complicated computations of the next section.

From (2.14), we have that in the case when $N = 1$, (2.15) is given by

$$\frac{d_1}{2} - \frac{d_3}{4}(u) + \frac{d_5}{16}(u'' + 3u^2) = 0. \qquad (4.1)$$

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Suppose d_1, d_3, and d_5 are given real numbers, and suppose $u \in L^2$ is a real-valued solution of the ordinary differential equation (4.1) (in the sense of distributions). Then from (4.1) we see that d_5 must be nonzero, and by dividing by d_5 if necessary, we can assume that $d_5 = 1$. Also, multiplying both sides of (4.1) by a test function $\phi_\tau(x) = \phi(x - \tau)$, where $\int_\mathbf{R} \phi(x)\ dx = 1$, and letting $\tau \to \infty$, we arrive at the conclusion that

$$\lim_{\tau \to \infty} \int_\mathbf{R} u\phi_\tau'' = \lim_{\tau \to \infty} \int_\mathbf{R} u''\phi_\tau = \lim_{\tau \to \infty} \int_\mathbf{R} (-3u^2 + 4d_3 u - 8d_1)\phi_\tau = -8d_1, \quad (4.2)$$

from which it follows that $d_1 = 0$. Letting $C = d_3$, we can then write (4.1) as

$$C\left(\frac{-u}{4}\right) + \frac{1}{16}(u'' + 3u^2) = 0. \tag{4.3}$$

Lemma 4.1. *Suppose $u \in L^2$ is a solution of (4.3) in the sense of distributions. Then u must be in H^s for all $s \geq 0$, and u is analytic on $\mathbf{R}$.*

Proof. Equation (4.3) can be rewritten as

$$u - u'' = au + bu^2, \tag{4.4}$$

where a and b are constants. Let $\mathcal{F}$ denote the Fourier transform, defined for $f \in L^1$ by $\mathcal{F}f(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}\ dx$, and extended to L^2 in the usual way. Letting $f = au$ and $g = bu^2$, taking the Fourier transform of (4.4), and dividing both sides by $(1 + k^2)^{1/2}$, we obtain

$$(1 + k^2)^{1/2}\mathcal{F}u = \frac{1}{(1 + k^2)^{1/2}}(\mathcal{F}f + \mathcal{F}g).$$

Since f is in L^2, then $\mathcal{F}f$ and $\mathcal{F}f/(1 + k^2)^{1/2}$ are in L^2; and since g is in L^1, then $\mathcal{F}g$ is bounded and continuous, and $\mathcal{F}g/(1 + k^2)^{1/2}$ is in L^2. Therefore $(1 + k^2)^{1/2}\mathcal{F}u \in L^2$, so $u \in H^1$. But it then follows that $u^2 \in L^2$, whence both f and g are in L^2, so (4.4) gives $u'' \in L^2$ and $u \in H^2$. Taking derivatives of (4.4) successively now easily gives that all higher-order derivatives of u are in L^2, so that $u \in H^s$ for all $s \geq 0$.

As a particular consequence, we have that u is a classical solution of (4.4) on $\mathbf{R}$. Therefore the fact that u is analytic on $\mathbf{R}$ follows from the fundamental theorems of ordinary differential equations, given that the right-hand side of (4.4) is an analytic function of u (see, e.g., section 1.8 of [5]). $\qquad\square$

Theorem 4.2. *Suppose $C \in \mathbf{R}$, and suppose $u \in L^2$ is a real-valued solution of (4.3), in the sense of distributions. Suppose also that u is not identically zero. Then $C > 0$, and there exists $K \in \mathbf{R}$ such that*

$$u = \frac{2C}{\cosh^2(\sqrt{C}x + K)}. \tag{4.5}$$

Remark. We can also write (4.5) in the form

$$u = \psi^{(1)}(x; a; \sqrt{C}),$$

where $a = e^{-2K}$.

Proof. Suppose u is an L^2 solution of (4.3) on $\mathbf{R}$, and is not identically 0. By Lemma 4.1, u is analytic on $\mathbf{R}$, and is in H^s for every $s \geq 0$. Hence, in particular, u and all its derivatives tend to zero as $|x| \to \infty$.

Multiplying (4.3) by u' and integrating gives

$$(u')^2 = 4Cu^2 - 2u^3, \tag{4.6}$$

where we have used the fact that $u \to 0$ and $u' \to 0$ as $x \to \infty$ to evaluate the constant of integration as zero. Letting $\zeta = -C + u/2$, we can rewrite (4.6) as

$$(\zeta')^2 = -4\zeta(\zeta + C)^2. \tag{4.7}$$

Since u is analytic on $\mathbf{R}$, then so is ζ. We know that ζ cannot be identically equal to $-C$ on $\mathbf{R}$, because u is not identically zero. Also, if $C \neq 0$, then ζ cannot be identically equal to 0 on $\mathbf{R}$, because this would contradict the fact that $u \to 0$ as $x \to \infty$. Therefore the set $\{x \in \mathbf{R} : \zeta(x) = 0 \text{ or } \zeta = -C\}$ must consist of isolated points (or be empty). Hence there exists an open interval I in $\mathbf{R}$ such that for all $x \in I$, $\zeta(x) \neq 0$ and $\zeta(x) + C \neq 0$. Also, from (4.7) it follows that $\zeta(x) < 0$ for $x \in I$.

Define Ω to be the domain in the complex plane given by

$$\Omega = \mathbf{C} - \{z : \mathfrak{Re}\, z \geq 0 \text{ and } \mathfrak{Im}\, z = 0\}. \tag{4.8}$$

Henceforth, for $z \in \Omega$ we will denote by $\sqrt{z}$ the branch of the square root function given by $\sqrt{z} = \sqrt{r}e^{i\theta/2}$ when $z = re^{i\theta}$ with $r > 0$ and $0 < \theta < 2\pi$. Thus $\sqrt{z}$ is an analytic function on Ω, and since $\zeta(x)$ takes values in Ω, then $\sqrt{\zeta(x)}$ is an analytic function of x on I.

From (4.7) we have that there exists a function $\theta : I \to \{-1, 1\}$ such that

$$\zeta' = 2i\theta(x)\sqrt{\zeta}(\zeta + C) \tag{4.9}$$

for all $x \in I$. Since $\zeta(x) + C \neq 0$ for all $x \in I$, it then follows from (4.9) that θ is analytic on I, and, since θ takes values in $\{-1, 1\}$, θ must therefore be constant on I.

Now define $v = -i\theta\sqrt{\zeta}$ on I, noting for future reference that, since $\zeta < 0$ on I, then v is real-valued. Let $\alpha = \sqrt{C}$. We then have from (4.9) that

$$\frac{v'}{C - v^2} = \frac{v'}{\alpha^2 - v^2} = 1. \tag{4.10}$$

To integrate (4.10), we first fix $x_0 \in I$, let $V = v(x_0)$, and define

$$L_{\alpha,V}(z) = \int_V^z \frac{dw}{\alpha^2 - w^2}.$$

Since $V \neq \pm\alpha$, this defines $L_{\alpha,V}$ as a single-valued, analytic function of z in some neighborhood of V. By shrinking I if necessary, we may assume that $L_{\alpha,V}(v(x))$ is defined for all $x \in I$, and so (4.10) may be integrated to give

$$L_{\alpha,V}(v(x)) = x - x_0 \tag{4.11}$$

for $x \in I$.

Our next goal will be to solve (4.11) for $v(x)$. Once this has been done, we can recover u from the formula

$$u = 2(\zeta + C) = 2(\alpha^2 - v^2). \tag{4.12}$$

Consider first the case when $\alpha \neq 0$ (and hence $C \neq 0$). In this case, by choosing an appropriate branch of the complex logarithm function, we could express $L_{\alpha,V}(z)$ as

$$L_{\alpha,V}(z) = \frac{1}{2\alpha}\left(\log\left(\frac{\alpha + z}{\alpha + V}\right) - \log\left(\frac{\alpha - z}{\alpha - V}\right)\right). \tag{4.13}$$

However, this will not be necessary, since we really only need to use the fact that

$$\exp(2\alpha L_{\alpha,V}(z)) = \left(\frac{\alpha + z}{\alpha - z}\right)\left(\frac{\alpha - V}{\alpha + V}\right) \tag{4.14}$$

for all z in some neighborhood of V. To see that (4.14) is true, define $f_1(z)$ to be the function on the left side of (4.14), and $f_2(z)$ to be the function on the right side. Then both f_1 and f_2 satisfy the differential equation $df/dz = (2\alpha/(\alpha^2 - z^2))f(z)$ in some neighborhood of V, and both take the value 1 at $z = V$. Since a solution f of the differential equation with a prescribed value at V is unique on any neighborhood of V where it is defined, f_1 must equal f_2 on some neighborhood of V.

Now multiplying both sides of (4.11) by 2α, taking exponentials, and using (4.14), one obtains

$$\frac{\alpha + v}{\alpha - v} = \left(\frac{\alpha + V}{\alpha - V}\right)e^{2\alpha(x - x_0)} = e^{2A}, \tag{4.15}$$

where

$$A = \alpha(x - x_0) + M \tag{4.16}$$

and M is any number such that

$$e^{2M} = \frac{\alpha + V}{\alpha - V}. \tag{4.17}$$

Solving (4.15) for v, we find that

$$v = \frac{y'}{y}$$

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where
$$y = \sinh A. \tag{4.18}$$

Substituting into (4.12), we find that

$$u = 2\left(\frac{\alpha^2 y^2 - (y')^2}{y^2}\right) = 2(y'/y)'. \tag{4.19}$$

Since u is analytic on $\mathbf{R}$, then the function on the right side of (4.19) is extendable to an analytic function on $\mathbf{R}$. This implies that y cannot have any zeroes on $\mathbf{R}$. We now have to determine the values of C for which this is possible. We consider separately the subcases in which $C > 0$ and $C < 0$.

If $C > 0$, then $\alpha = \sqrt{C}$ is real, and from (4.16) and (4.17) we see that we can take

$$A = \alpha x + K + i\sigma\pi/2, \tag{4.20}$$

where K is real and either $\sigma = 0$ or $\sigma = 1$, according to whether $(V + \alpha)/(V - \alpha)$ is positive or negative. If $\sigma = 0$, then

$$y = \sinh(\alpha x + K),$$

which equals zero for some $x \in \mathbf{R}$, so u has a singularity at this x. On the other hand, if $\sigma = 1$, then (4.18) gives

$$y = i\cosh(\alpha x + K),$$

which does not vanish at any point of $\mathbf{R}$. In this case the function u given by (4.19) is nonsingular, and in fact we recover the solution given by (4.5).

If, on the other hand, $C < 0$, then $\alpha = i\sqrt{|C|}$ is purely imaginary, so

$$\left|\frac{\alpha + V}{\alpha - V}\right| = 1.$$

It then follows from (4.16) and (4.17) that A is purely imaginary, and we can write

$$A = i(\sqrt{|C|}x + K), \tag{4.21}$$

where K is real. Then (4.18) gives

$$y = i\sin(\sqrt{|C|}x + K),$$

contradicting the fact that y cannot have any zeroes on $\mathbf{R}$. We conclude that C cannot be negative.

It remains to show that α and C cannot equal zero. For if they were, then integrating (4.10) would give $v = (x + K)^{-1}$, where K is a constant, and so by (4.12),

$$u = \frac{-2}{(x + K)^2}$$

for all $x \in I$. But this contradicts the fact that u is analytic on all of $\mathbf{R}$.

We have now shown that if (4.3) has a solution in L^2 that is not identically zero, then C must be positive, and in that case the only solutions are those given by (4.5). So the proof is complete. $\square$

5 The stationary equation for $N = 2$

According to Theorem 3.8 and Definition 3.12, every N-soliton solution of the KdV hierarchy has profiles which are solutions of the stationary equation (2.15), or more specifically of (3.38). In this section, for the case $N = 2$, we prove a converse to this result: every H^2 solution of (2.15) with $N = 2$ must be either a 1-soliton profile or a 2-soliton profile.

Taking $N = 2$ in (2.15), we obtain from (2.14) the equation

$$\frac{d_1}{2} - \frac{d_3}{4}(u) + \frac{d_5}{16}(u'' + 3u^2) - \frac{d_7}{64}(u'''' + 5u'^2 + 10uu'' + 10u^3) = 0. \quad (5.1)$$

We may assume in what follows that $d_7 \neq 0$, for otherwise we are back in the case $N = 1$, which has already been handled in section 4. Dividing by d_7 if necessary, we can therefore take $d_7 = 1$ without losing generality. We may also henceforth assume that $d_1 = 0$, since a computation similar to that given in (4.2) shows that this must be the case if (5.1) has a solution u in H^2.

Lemma 5.1. *Suppose $u \in H^2$ is a solution of equation* (5.1) *in the sense of distributions. Then u must be in H^s for all $s \geq 0$, and u is analytic on* **R**. *In particular, we have*

$$\lim_{x \to \infty} u(x) = \lim_{x \to \infty} u'(x) = \lim_{x \to \infty} u''(x) = \lim_{x \to \infty} u'''(x) = 0. \quad (5.2)$$

Proof. Taking $d_1 = 0$ in (5.1) and solving for u'''', we obtain

$$u'''' = au + bu^2 + cu^3 + d(u')^2 + eu'' + fuu'', \quad (5.3)$$

where a, b, c, d, e, f are constants. Since $u \in H^2$, then all the terms on the right-hand side of (5.3) are in L^2, so $u'''' \in L^2$ as well. Hence $u \in H^4$, and this already yields (5.2). It also implies that u is a classical solution of (5.1), so by fundamental theorems of ordinary differential equations, u is analytic. Finally, taking derivatives of (5.3) successively and applying an inductive argument yields that $u \in H^s$ for all $s \geq 0$. $\square$

Theorem 5.2. *Suppose $d_1 = 0$, $d_7 = 1$, and d_3 and d_5 are arbitrary real numbers, and suppose $u \in H^2$ is a nontrivial (i.e., not identically zero) distribution solution of equation* (5.1).

Then either
(i) u is a 1-soliton profile given by

$$u = \psi^{(1)}\left(x; a; \sqrt{C}\right), \quad (5.4)$$

where C is a positive root of the quadratic equation

$$z^2 - d_5 z + d_3 = 0. \quad (5.5)$$

and a is a real number such that a > 0; or
(ii) u is a 2-soliton profile given by

$$u = \psi^{(2)}\left(x; a_1, a_2; \sqrt{C_1}, \sqrt{C_2}\right), \tag{5.6}$$

where C_1 and C_2 are roots of equation (5.5) with $0 < C_1 < C_2$, and a_1 and a_2 are real numbers such that $a_2 < 0 < a_1$.

Proof. Suppose u is a nontrivial distribution solution of (5.1) with $d_7 = 1$ and $d_1 = 0$. By Lemma 5.1 we may assume that u is analytic on $\mathbf{R}$ and satisfies (5.2).

Following Chapter 12 of [7] we define, for $x \in \mathbf{R}$ and $\zeta \in \mathbf{C}$,

$$\hat{R}(x, \zeta) = \hat{R}_0 + \hat{R}_1\zeta + \hat{R}_2\zeta^2, \tag{5.7}$$

where

$$\hat{R}_0 = d_3 R_1 + d_5 R_3 + d_7 R_5 = \frac{d_3}{2} - \frac{d_5}{4} u + \frac{1}{16}(u'' + 3u^2)$$

$$\hat{R}_1 = d_5 R_1 + d_7 R_3 = \frac{d_5}{2} - \frac{1}{4} u \tag{5.8}$$

$$\hat{R}_2 = d_7 R_1 = \frac{1}{2}.$$

We claim that

$$\hat{R}''' + 4u\hat{R}' + 2u'\hat{R} + 4\zeta\hat{R}' = 0. \tag{5.9}$$

Indeed, substituting (5.7) into (5.9) and using Lemma 2.2, we find that the left side of (5.9) is equal to

$$-4(d_3 R_3' + d_5 R_5' + d_7 R_7') +$$
$$+ \zeta(d_5 Q_1 + d_7 Q_3 + 4d_3 R_1') + \zeta^2(d_7 Q_1 + 4d_5 R_1') + \zeta^3(4d_7 R_1'), \tag{5.10}$$

where $Q_1 = R_1''' + 4uR_1' + 2u'R_1 + 4R_3'$ and $Q_3 = R_3''' + 4uR_3' + 2u'R_3 + 4R_5'$. But since u is a solution of (5.1) and $d_1 = 0$, we have that $d_3 R_3 + d_5 R_5 + d_7 R_7 = 0$, so the first term in (5.10) vanishes, and Q_1 and Q_3 are zero by virtue of Lemma 2.2. Since $R_1' = 0$, this proves (5.9).

Multiplying (5.9) by $\hat{R}$ and integrating with respect to x gives

$$2\hat{R}''\hat{R} - \hat{R}'^2 + 4(u + \zeta)\hat{R}^2 = P(\zeta), \tag{5.11}$$

where $P(\zeta)$ is a polynomial in ζ with coefficients that are independent of x. From (2.14), (5.2), and (5.8) we see that

$$\lim_{x \to \infty} \hat{R}_0 = d_3/2$$

$$\lim_{x \to \infty} \hat{R}_1 = d_5/2 \tag{5.12}$$

$$\lim_{x \to \infty} \hat{R}_2 = d_7/2 = 1/2.$$

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Also,
$$\lim_{x \to \infty} \hat{R}'_i(x) = \lim_{x \to \infty} \hat{R}''_i(x) = 0 \quad \text{for } i = 0, 1, 2.$$

Therefore, taking the limit of (5.11) as $x \to \infty$, we get that
$$P(\zeta) = \zeta(d_3 + d_5\zeta + \zeta^2)^2. \tag{5.13}$$

Combining (5.11) and (5.13) gives
$$2\hat{R}''\hat{R} - \hat{R}'^2 + 4(u + \zeta)\hat{R}^2 = \zeta(d_3 + d_5\zeta + \zeta^2)^2. \tag{5.14}$$

Let C_1 and C_2 denote the (possibly repeated) roots of equation (5.5). As roots of a polynomial with real coefficients, C_1 and C_2 are either both real numbers or are complex conjugates of each other, and we may assume they are ordered so that $\mathfrak{Re}\, C_1 \leq \mathfrak{Re}\, C_2$. Then
$$d_3 = C_1 C_2 \quad \text{and} \quad d_5 = C_1 + C_2, \tag{5.15}$$

and so, by (5.14),
$$2\hat{R}''\hat{R} - \hat{R}'^2 + 4(u + \zeta)\hat{R}^2 = \zeta(\zeta + C_1)^2(\zeta + C_2)^2. \tag{5.16}$$

Let us now view the function $\hat{R}(x, \zeta)$ as a polynomial in the complex variable ζ with coefficients which are analytic functions of x. Our next goal is to study the roots of this polynomial.

First, observe that since $\hat{R}_0$ is, like u, analytic on $\mathbf{R}$, then $\hat{R}_0$ is either identically zero on $\mathbf{R}$ or has only isolated zeros. But if $\hat{R}_0$ is identically zero on $\mathbf{R}$, then by (5.12) we must have $d_3 = 0$. The equation $\hat{R}_0 = 0$ in (5.8) is then seen to take the form of (4.3), with C replaced by d_5, and so it follows from Theorem 4.2 that $d_5 > 0$ and u is given by (5.4). Notice also that since $d_3 = 0$, d_5 is a positive root of (5.5). Thus statement (i) of the Theorem holds in this case. Therefore, we can, without loss of generality, assume that $\hat{R}_0$ has only isolated zeros, and hence there exists an open interval $I \subseteq \mathbf{R}$ such that $\hat{R}_0(x) \neq 0$ for all $x \in I$. It then follows that for all $x \in I$, $\zeta = 0$ is not a root of $\hat{R}(x, \zeta)$.

We claim that there exists at least one $x_0 \in I$ such that the polynomial $\hat{R}(x_0, \zeta)$ has distinct roots ζ_1 and ζ_2. For if this is not the case, then there exists a function $\zeta_1(x)$ such that for all $x \in I$,
$$\hat{R}(x, \zeta) = \frac{1}{2}(\zeta - \zeta_1(x))^2. \tag{5.17}$$

From (5.17) and (5.7) we have that $\zeta_1(x)^2 = 2\hat{R}_0(x)$, and since $\hat{R}_0(x)$ is nonzero for all $x \in I$ it follows that $\zeta_1(x)$ is analytic, and hence differentiable, as a function of x. Thus we can differentiate (5.17) with respect to x to obtain
$$\hat{R}'(x, \zeta) = -(\zeta - \zeta_1(x))\zeta_1'(x)$$

for $x \in I$. But then substituting $\zeta = \zeta_1(x)$ into (5.16) gives
$$0 = \zeta_1(x)(\zeta_1(x) + C_1)^2(\zeta_1(x) + C_2)^2.$$

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Since $\zeta_1(x) \neq 0$ for $x \in I$, it follows that for all $x \in I$, either $\zeta_1(x) = -C_1$ or $\zeta_1(x) = -C_2$. Since $\zeta_1(x)$ is analytic on I, the set of points where $\zeta_1(x)$ takes a given value must either be a discrete subset of I, or consist of all of I. Since the union of two discrete subsets of I cannot equal all of I, it must be that either $\zeta_1(x) \equiv -C_1$ on I or $\zeta_1(x) \equiv -C_2$ on I. Hence (5.17) gives, for either $j = 1$ or $j = 2$,

$$\hat{R}(x, \zeta) = \frac{1}{2}(\zeta^2 + 2C_j\zeta + C_j^2),$$

and so, by (5.7),

$$\hat{R}_1 = \frac{d_5}{2} - \frac{1}{4}u = C_j$$

holds for all $x \in I$. But this implies that u is constant on I, and since u is analytic on $\mathbf{R}$, then u must be constant on $\mathbf{R}$. Then (5.2) gives that u is identically zero, contrary to our assumption that u is nontrivial. Thus the claim has been proved.

It now follows from standard perturbation theory [21] that, by shrinking I if necessary to a smaller neighborhood, we can assume that there exist analytic, nonzero functions ζ_1 and ζ_2 on I such that $\zeta_1(x) \neq \zeta_2(x)$ and $\hat{R}(x, \zeta_1(x)) = \hat{R}(x, \zeta_2(x)) = 0$ for all $x \in I$. We therefore have

$$\hat{R}(x, \zeta) = \frac{1}{2}(\zeta - \zeta_1(x))(\zeta - \zeta_2(x)), \tag{5.18}$$

for all $x \in I$. Also, since ζ_1 and ζ_2 are roots of a real polynomial, we have that either ζ_1 and ζ_2 are both real on I or $\zeta_1^* = \zeta_2$ on I.

Our goal in what follows is to obtain a second-order system of differential equations for ζ_1 and ζ_2, which can then be integrated explicitly to find ζ_1 and ζ_2. Once this is accomplished, it is easy to recover u, since (5.7) and (5.18) imply that $\zeta_1 + \zeta_2 = -2\hat{R}_1$, and hence

$$u = 2(\zeta_1 + \zeta_2 + d_5). \tag{5.19}$$

For $i = 1, 2$, we have $\hat{R}(x, \zeta_i(x)) = 0$, and hence it follows from (5.16) that

$$\hat{R}'(x, \zeta_i(x))^2 = -\zeta_i(\zeta_i + C_1)^2(\zeta_i + C_2)^2. \tag{5.20}$$

Differentiating (5.18) with respect to x, we obtain

$$\hat{R}'(x, \zeta) = -\frac{1}{2}\left[(\zeta - \zeta_1)\zeta_2'(x) + (\zeta - \zeta_2)\zeta_1'(x)\right].$$

Therefore

$$\hat{R}'(x, \zeta_1(x)) = -\frac{1}{2}\left(\zeta_1(x) - \zeta_2(x)\right)\zeta_1'(x)$$

$$\hat{R}'(x, \zeta_2(x)) = -\frac{1}{2}\left(\zeta_2(x) - \zeta_1(x)\right)\zeta_2'(x).$$

From (5.20) we then have that

$$(\zeta_1 - \zeta_2)^2(\zeta_1')^2 = -4\zeta_1(\zeta_1 + C_1)^2(\zeta_1 + C_2)^2$$
$$(\zeta_1 - \zeta_2)^2(\zeta_2')^2 = -4\zeta_2(\zeta_2 + C_1)^2(\zeta_2 + C_2)^2. \tag{5.21}$$

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Note that, since C_1 and C_2 are either both real numbers, or are complex conjugates of one another, it follows from (5.21) that if ζ_1 and ζ_2 are real, they must necessarily take negative values at all points of I. Since ζ_1 and ζ_2 are nonzero functions on I, it follows that both ζ_1 and ζ_2 map I into the domain Ω in the complex plane defined in (4.8). Also, since $\zeta_1 - \zeta_2$ has no zeros in I, we have from (5.21) that

$$\zeta_1' = \frac{2i\theta_1(x)\sqrt{\zeta_1}(\zeta_1 + C_1)(\zeta_1 + C_2)}{\zeta_1 - \zeta_2}$$

$$\zeta_2' = \frac{2i\theta_2(x)\sqrt{\zeta_2}(\zeta_2 + C_1)(\zeta_2 + C_2)}{\zeta_1 - \zeta_2} \ ,$$

(5.22)

where $\theta_i(x) \in \{-1, 1\}$ for $i = 1, 2$. Here, as throughout the paper, we use $\sqrt{z}$ to denote the analytic branch of the square root function on Ω defined after (4.8).

The change of variables from u to (ζ_1, ζ_2), which reduces the stationary equation (5.1) to the separable system (5.22), is due originally to Dubrovin in [10] (see also [12, 20, 29], and chapter 12 of [7]). These authors use the same change of variables (or, more precisely, its generalization to the case of general N) to, among other things, determine the time evolution of finite-gap solutions of the Korteweg-de Vries hierarchy.

Again using the analyticity of ζ_1 and ζ_2, and taking I smaller if necessary, we can reduce consideration to the following two cases: either there exist $i, j \in \{1, 2\}$ such that

$$\zeta_i(x) + C_j = 0 \quad \text{for all } x \in I, \tag{5.23}$$

or, for all $i, j \in \{1, 2\}$,

$$\zeta_i(x) + C_j \neq 0 \quad \text{for all } x \in I. \tag{5.24}$$

Suppose (5.23) holds, with for example $i = 1$ and $j = 1$; the argument for other choices of i and j is exactly similar. Then from (5.22) we obtain

$$\zeta_2' = -2i\theta_2(x)\sqrt{\zeta_2}(\zeta_2 + C_2). \tag{5.25}$$

We know that ζ_2 is not identically equal to $-C_2$ on I, for otherwise (5.15), (5.19), and $\zeta_1 \equiv -C_1$ would imply that u is identically equal to 0 on I, and hence also on $\mathbf{R}$. Therefore, by taking I smaller if necessary, we may assume that ζ_2 is never equal to $-C_2$ on I.

It then follows from (5.25) that $\theta_2(x)$ is analytic on I. But since θ_2 takes values in $\{-1, 1\}$, the only way this can happen is if θ_2 is constant on I. Setting $v = i\theta_2\sqrt{\zeta_2}$ in (5.25), we obtain

$$\frac{v'}{C_2 - v^2} = 1,$$

which is equation (4.10) with C replaced by C_2.

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Moreover, we also know that C_1 and C_2 must be real, since otherwise we would have $C_2 = C_1^*$, and together with $\zeta_1 \equiv -C_1$ and $\zeta_1 = \zeta_2^*$ this would imply $\zeta_2 \equiv -C_2$ on I. Since C_1 is real, then $\zeta_1 \equiv -C_1$ implies that ζ_1 is real, so ζ_2 must also be real on I. Since we also know that ζ_2 is never equal to zero on I, it follows from (5.21) that $\zeta_2 < 0$ on I. Hence v is real-valued on I. Therefore we can reprise the proof of Theorem 4.2 from (4.10) onwards, replacing C by C_2 throughout (notice that by (5.19) we have $u = 2(\zeta_2 + C_2) = 2(C_2 - v^2)$ in this case as well). The conclusion is that $C_2 > 0$ and that

$$u = \frac{2C_2}{\cosh^2(\sqrt{C_2}x + K)}.$$

In light of the remark following Theorem 4.2, we obtain the formula (5.4) for u, and this must hold on the entire line. Thus the statement of the theorem is proved in this case.

Therefore we may assume henceforth that (5.24) holds for all $i, j \in \{1, 2\}$. In this case, the right-hand sides of both equations in (5.22) are never zero on I, so as in the preceding cases it follows from (5.22) that θ_1 and θ_2 are both analytic and hence constant functions on I, with value either -1 or 1. Set

$$\begin{aligned}
v_1 &= -i\theta_1\sqrt{\zeta_1} \\
v_2 &= i\theta_2\sqrt{\zeta_2},
\end{aligned} \tag{5.26}$$

and define α_j, for $j = 1, 2$, to be complex numbers such that

$$\alpha_j^2 = C_j. \tag{5.27}$$

For definiteness we will choose α_j to be the square root of C_j given by $\alpha_j = |C_j|^{1/2}e^{i\theta/2}$, where $-\pi < \theta \le \pi$ and $C_j = |C_j|e^{i\theta}$. In particular, this choice guarantees that if $C_1^* = C_2$, then $\alpha_1^* = \alpha_2$.

Since ζ_1 and ζ_2 are nonzero on I, so are v_1 and v_2. Also, as noted above after (5.21), either ζ_1 and ζ_2 are both negative at all points of I, or $\zeta_1^* = \zeta_2$ on I. In the former case, we have that v_1 and v_2 are real-valued on I. In the latter case, we see by taking the conjugate of the first equation in (5.22), comparing the result to the second equation in (5.22), and using the fact that $\sqrt{z^*} = -(\sqrt{z})^*$ on Ω, that $\theta_1 = -\theta_2$ on I. Therefore from (5.26) we obtain that $v_1^* = v_2$ on I.

We can now rewrite (5.22) as the following system for v_1 and v_2:

$$\begin{aligned}
v_1' &= \frac{(\alpha_1^2 - v_1^2)(\alpha_2^2 - v_1^2)}{v_2^2 - v_1^2} \\
v_2' &= \frac{(\alpha_1^2 - v_2^2)(\alpha_2^2 - v_2^2)}{v_1^2 - v_2^2},
\end{aligned} \tag{5.28}$$

where either v_1 and v_2 are both real-valued on I, or $v_1^* = v_2$ on I. Choose $x_0 \in I$, and define

$$\begin{aligned}
V_1 &= v_1(x_0) \\
V_2 &= v_2(x_0).
\end{aligned} \tag{5.29}$$

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(For future reference we note that, as values of v_1 and v_2, V_1 and V_2 must both be nonzero, and either V_1 and V_2 both real or $V_1^* = V_2$.) Recalling that $v_1^2 \neq v_2^2$ on I, we have that the right-hand sides of the equations in (5.28) define analytic functions of v_1 and v_2 on I. Therefore the system (5.28), together with the initial data (5.29), uniquely determines v_1 and v_2 on some neighborhood of x_0 on I. Furthermore, from v_1 and v_2 one can then recover u via (5.19) as

$$u(x) = 2(-v_1^2(x) - v_2^2(x) + \alpha_1^2 + \alpha_2^2). \tag{5.30}$$

We will complete the proof of Theorem 5.2 by explicitly solving the initial-value problem (5.28) and (5.29) for v_1 and v_2, and then showing that, of the functions u which arise from these solutions via (5.30), the only ones which extend to H^1 functions on $\mathbf{R}$ are those given by (5.6).

The system (5.28) can be integrated by separating the variables v_1 and v_2. Since (5.24) holds on I, we have that $v_i(x) \neq \alpha_j$ on I for $i, j \in \{1, 2\}$. This allows us to rewrite (5.28) in the form

$$\frac{v_1'}{(\alpha_1^2 - v_1^2)(\alpha_2^2 - v_1^2)} + \frac{v_2'}{(\alpha_1^2 - v_2^2)(\alpha_2^2 - v_2^2)} = 0$$

$$\frac{-v_1^2 v_1'}{(\alpha_1^2 - v_1^2)(\alpha_2^2 - v_1^2)} + \frac{-v_2^2 v_2'}{(\alpha_1^2 - v_2^2)(\alpha_2^2 - v_2^2)} = 1. \tag{5.31}$$

To compute the solutions of (5.31) in the cases when the quantities α_i^2 coincide or are zero, it will be helpful to consider as well a system in which the values of the constants α_i are slightly perturbed. For any positive number δ and complex number z_0, let $B_\delta(z_0)$ denote the open ball of radius δ centered at z_0 in $\mathbf{C}$. Choose δ to be any positive number such that $\delta < \frac{1}{4} \min\{|\alpha_i - V_j| : i, j = 1, 2\}$ and $\delta < |V_1 - V_2|$. For each $j \in \{1, 2\}$, we define functions $G_j = G_j(\beta_1, \beta_2, v)$ and $H_j = H_j(\beta_1, \beta_2, v)$ on $B_\delta(\alpha_1) \times B_\delta(\alpha_2) \times B_\delta(V_j)$ by

$$G_j(\beta_1, \beta_2, v) = \int_{V_j}^{v} \frac{dw}{(\beta_1^2 - w^2)(\beta_2^2 - w^2)}$$

$$H_j(\beta_1, \beta_2, v) = \int_{V_j}^{v} \frac{-w^2 dw}{(\beta_1^2 - w^2)(\beta_2^2 - w^2)} . \tag{5.32}$$

From the definition of δ we know that the integrands in (5.32) are nonsingular functions of w on $B_\delta(V_j)$, so G_j and H_j are well-defined and are analytic functions of β_1, β_2, and v on their domains, as long as the integrals in their definition are taken over paths from V_j to v which lie within $B_\delta(V_j)$. We can then define $F :$ $\mathbf{C}^2 \times \mathbf{C}^2 \times \mathbf{R} \to \mathbf{C}^2$, with domain $U_F = B_\delta(\alpha_1) \times B_\delta(\alpha_2) \times B_\delta(V_1) \times B_\delta(V_2) \times \mathbf{R}$, by setting

$$F(\beta_1, \beta_2, v_1, v_2, x) = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} G_1(\beta_1, \beta_2, v_1) + G_2(\beta_1, \beta_2, v_2) \\ H_1(\beta_1, \beta_2, v_1) + H_2(\beta_1, \beta_2, v_2) - x + x_0 \end{bmatrix}. \tag{5.33}$$

Lemma 5.3. *There exist numbers $\delta_1 \in (0, \delta)$ and $\delta_2 > 0$ such that for each (β_1, β_2, x) in $B_{\delta_1}(\alpha_1) \times B_{\delta_1}(\alpha_2) \times \{|x - x_0| < \delta_1\} \subseteq \mathbf{C}^2 \times \mathbf{R}$, there is a unique pair (v_1, v_2) in $B_{\delta_2}(V_2) \times B_{\delta_2}(V_2) \subseteq \mathbf{C}^2$ satisfying*

$$F(\beta_1, \beta_2, v_1, v_2, x) = 0. \tag{5.34}$$

The functions $v_1(\beta_1, \beta_2, x)$ and $v_2(\beta_1, \beta_2, x)$ so defined are analytic functions of their arguments, and for each $(\beta_1, \beta_2) \in B_{\delta_1}(\alpha_1) \times B_{\delta_1}(\alpha_2)$, are solutions of the system of ordinary differential equations

$$\frac{v_1'}{(\beta_1^2 - v_1^2)(\beta_2^2 - v_1^2)} + \frac{v_2'}{(\beta_1^2 - v_2^2)(\beta_2^2 - v_2^2)} = 0$$

$$\frac{-v_1^2 v_1'}{(\beta_1^2 - v_1^2)(\beta_2^2 - v_1^2)} + \frac{-v_2^2 v_2'}{(\beta_1^2 - v_2^2)(\beta_2^2 - v_2^2)} = 1, \tag{5.35}$$

on $\{|x - x_0| < \delta_1\}$, with initial conditions

$$v_1(\beta_1, \beta_2, x_0) = V_1$$
$$v_2(\beta_1, \beta_2, x_0) = V_2. \tag{5.36}$$

Proof. A computation of the Jacobian of F with respect to the variables v_1 and v_2, reveals that, at all points $P = (\beta_1, \beta_2, v_1, v_2, x)$ in the domain U_F of F, we have

$$\nabla_{v_1, v_2} F(P) = \begin{bmatrix} \partial F_1/\partial v_1 & \partial F_1/\partial v_2 \\ \partial F_2/\partial v_1 & \partial F_2/\partial v_2 \end{bmatrix}\bigg|_P$$

$$= \begin{bmatrix} \dfrac{1}{(\beta_1^2 - v_1^2)(\beta_2^2 - v_1^2)} & \dfrac{1}{(\beta_1^2 - v_2^2)(\beta_2^2 - v_2^2)} \\ \dfrac{-v_1^2}{(\beta_1^2 - v_1^2)(\beta_2^2 - v_1^2)} & \dfrac{-v_2^2}{(\beta_1^2 - v_2^2)(\beta_2^2 - v_2^2)} \end{bmatrix}. \tag{5.37}$$

The determinant of the matrix in (5.37) is $(v_1^2 - v_2^2)/\prod_{i,j=1,2}(\beta_i^2 - v_j^2)$, and is therefore nonzero for all $P \in U_F$. In particular, when $P_0 = (\alpha_1, \alpha_2, V_1, V_2, x_0)$ we have that $\nabla_{v_1, v_2} F(P_0)$ is an invertible map from $\mathbf{C}^2$ to $\mathbf{C}^2$. Moreover, $F(P_0) = 0$. The assertions of the Lemma concerning the existence, uniqueness, and analyticity of the functions v_1 and v_2 which satisfy (5.34) therefore follow from the Implicit Function Theorem (cf. §15 of [6]). Equations (5.35) then follow by differentiating both sides of (5.34) with respect to x. The initial conditions (5.36) are a consequence of the uniqueness assertion for the v_i and the fact that

$$F(\beta_1, \beta_2, V_1, V_2, x_0) = 0$$

for each $(\beta_1, \beta_2) \in B_{\delta_1}(\alpha_1) \times B_{\delta_1}(\alpha_2)$. $\qquad \square$

Motivated by (5.30), we now define, for each $(\beta_1, \beta_2, x) \in B_{\delta_1}(\alpha_1) \times B_{\delta_1}(\alpha_2) \times \{|x - x_0| < \delta_1\}$,

$$u(\beta_1, \beta_2, x) = 2(-v_1(\beta_1, \beta_2, x)^2 - v_2(\beta_1, \beta_2, x)^2 + \beta_1^2 + \beta_2^2). \tag{5.38}$$

Corollary 5.4. *The solution u of* (5.1) *described in Theorem 5.2 is related to the functions $u(\beta_1, \beta_2, x)$ by*

$$u(x) = u(\alpha_1, \alpha_2, x) = \lim_{(\beta_1, \beta_2) \to (\alpha_1, \alpha_2)} u(\beta_1, \beta_2, x) \qquad (5.39)$$

for all x such that $|x - x_0| < \delta_1$.

Proof. By Lemma 5.3, the functions $v_1(\alpha_1, \alpha_2, x)$ and $v_2(\alpha_1, \alpha_2, x)$ satisfy (5.31) for $|x - x_0| < \delta_1$, and therefore, since $\nabla_{v_1, v_2} F(P_0)$ is invertible, also satisfy (5.28). Comparing the initial conditions (5.36) and (5.29), we see from the uniqueness of the solutions of the initial value problem for (5.28) that $(v_1(x), v_2(x)) = (v_1(\alpha_1, \alpha_2, x), v_2(\alpha_1, \alpha_2, x))$ for all x in some neighborhood of x_0. Putting $(\beta_1, \beta_2) = (\alpha_1, \alpha_2)$ in (5.38) and comparing with (5.30), we see that $u(x)$ then agrees with $u(\alpha_1, \alpha_2, x)$ on this neighborhood. Since $u(x)$ is analytic on **R** and $u(\alpha_1, \alpha_2, x)$ is analytic on $\{|x - x_0| < \delta_1\}$, they must agree on all of $\{|x - x_0| < \delta_1\}$. Finally, the assertion that $u(x)$ is given by the limit in (5.39) follows from the fact that $u(\beta_1, \beta_2, x)$ is analytic and hence continuous in β_1 and β_2. $\qquad\square$

The next step in the proof of Theorem 5.2 is to explicitly determine the functions $u(\beta_1, \beta_2, x)$ defined in (5.38). Generically, we will have that β_1^2 and β_2^2 are distinct and nonzero, even if this is not true when $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2$. In this case, we have the following Lemma.

Lemma 5.5. *Suppose $(\beta_1, \beta_2, x) \in B_{\delta_1}(\alpha_1) \times B_{\delta_1}(\alpha_2) \times \{|x - x_0| < \delta_1\}$, and suppose that $\beta_1^2 \neq \beta_2^2$, and $\beta_j \neq 0$ for $j = 1, 2$. Then*

$$u(\beta_1, \beta_2, x) = 2(D'/D)', \qquad (5.40)$$

where

$$D = D(\beta_1, \beta_2, x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \qquad (5.41)$$

with

$$y_j = \sinh(\beta_j(x - x_0) + M_j) \qquad (5.42)$$

for $j = 1, 2$. Here M_1 and M_2 can be taken to be any complex numbers satisfying

$$e^{2M_j} = \frac{(\beta_j + V_1)(\beta_j + V_2)}{(\beta_j - V_1)(\beta_j - V_2)} \qquad (5.43)$$

for $j = 1, 2$.

Proof. Under the stated assumptions on (β_1, β_2, x), we have, for $v \in B_\delta(V_j)$,

$$G_j(\beta_1, \beta_2, v) = \frac{1}{\beta_2^2 - \beta_1^2} \left[L_{\beta_1, V_j}(v) - L_{\beta_2, V_j}(v) \right]$$

$$H_j(\beta_1, \beta_2, z) = \frac{1}{\beta_2^2 - \beta_1^2} \left[-\beta_1^2 L_{\beta_1, V_j}(v) + \beta_2^2 L_{\beta_2, V_j}(v) \right],$$

$$(5.44)$$

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where $L_{\beta,V}$ is defined in (4.13). Let v_j denote $v_j(\beta_1, \beta_2, x)$ for $j = 1, 2$, and define

$$w_j = 2\beta_j L_{\beta_j, V_1}(v_1) + 2\beta_j L_{\beta_j, V_2}(v_2). \qquad (5.45)$$

Substituting (5.44) into (5.33), we see that (5.34) can be rewritten as a linear system for w_1 and w_2:

$$\beta_2 w_1 - \beta_1 w_2 = 0$$
$$-\beta_1 w_1 + \beta_2 w_2 = 2(\beta_2^2 - \beta_1^2)(x - x_0).$$

Since $\beta_1 \neq \beta_2$, the system has a unique solution, given by

$$w_j = 2\beta_j(x - x_0) \quad \text{for } j = 1, 2. \qquad (5.46)$$

Now from (4.14) and (5.45) we have that

$$e^{w_j} = \left(\frac{\beta_j + v_1}{\beta_j - v_1}\right)\left(\frac{\beta_j + v_2}{\beta_j - v_2}\right)\left(\frac{\beta_j - V_1}{\beta_j + V_1}\right)\left(\frac{\beta_j - V_2}{\beta_j + V_2}\right).$$

Therefore after exponentiating both sides of (5.46), we obtain, for $j = 1, 2$,

$$\frac{(\beta_j + v_1)(\beta_j + v_2)}{(\beta_j - v_1)(\beta_j - v_2)} = \frac{(\beta_j + V_1)(\beta_j + V_2)}{(\beta_j - V_1)(\beta_j - V_2)} \, e^{2\beta_j(x-x_0)} = e^{2(\beta_j(x-x_0)+M_j)}, \quad (5.47)$$

where M_1 and M_2 are any complex numbers such that (5.43) holds.

If we now define

$$W_1 = -(v_1 + v_2)$$
$$W_0 = v_1 v_2,$$

then (5.47) can be rewritten as

$$\begin{bmatrix} y_1 & y_1' \\ y_2 & y_2' \end{bmatrix} \begin{bmatrix} W_0 \\ W_1 \end{bmatrix} = -\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix}, \qquad (5.48)$$

where y_j is given by (5.42) for $j = 1, 2$. Solving (5.48) by Cramer's rule, we find that W_1 and W_0 are given by (3.6) and (3.7), with $N = 2$. Therefore we can conclude from Corollary 3.4 that

$$W_1' = W_1^2 - 2W_0 - \beta_1^2 - \beta_2^2. \qquad (5.49)$$

On the other hand, from (5.38) we have

$$u(\beta_1, \beta_2, x) = 2(2W_0 - W_1^2 + \beta_1^2 + \beta_2^2), \qquad (5.50)$$

and from (5.49) and (5.50) it then follows that $u(\beta_1, \beta_2, x) = -2W_1'$. By (3.6), we have $W_1' = -D'/D$, with D given by (5.41). Then (5.40) follows. $\qquad \square$

We now determine the conditions on β_1 and β_2 under which (5.40) defines a nonsingular solution on $\mathbf{R}$, or in other words, under which $D(\beta_1, \beta_2, x)$ has no zeroes on $\mathbf{R}$.

Lemma 5.6. *Suppose $(\beta_1, \beta_2) \in B_{\delta_1}(\alpha_1) \times B_{\delta_1}(\alpha_2)$, and suppose that $\beta_1^2 \neq \beta_2^2$, and $\beta_j \neq 0$ for $j = 1, 2$. Suppose also that $0 \leq \mathfrak{Re}\, \beta_1 \leq \mathfrak{Re}\, \beta_2$, and either β_1^2 and β_2^2 are both real numbers, or $\beta_1^* = \beta_2$.*

If, as a function of x, $u(\beta_1, \beta_2, x)$ can be analytically continued to an analytic function on the entire real line, then $0 < \beta_1 < \beta_2$, and there exist numbers a_1 and a_2 with $a_2 < 0 < a_1$ such that

$$u(\beta_1, \beta_2, x) = \psi^{(2)}(x; a_1, a_2; \beta_1, \beta_2)$$

on $\mathbf{R}$.

Proof. Observe that, according to Lemma 5.5, if $u(\beta_1, \beta_2, x)$ can be analytically continued to an analytic function on the real line, then the function $D(\beta_1, \beta_2, x)$ defined in (5.41)–(5.43) must be nonzero for all $x \in \mathbf{R}$. In fact, from (5.40) it is easy to see that, at any point x where $D(\beta_1, \beta_2, x)$ has a zero, $u(\beta_1, \beta_2, x)$ will have a pole of order two.

If $\beta_1^* = \beta_2$, then since either V_1 and V_2 are both real or $V_1^* = V_2$, we see from (5.43) that we can choose M_1 and M_2 so that $M_1^* = M_2$. Define $K_j = -\beta_j x_0 + M_j$ and

$$A_j = \beta_j(x - x_0) + M_j = \beta_j x + K_j,$$

so that $A_1^* = A_2$ and, by (5.42), $y_j = \sinh A_j$ for $j = 1, 2$. Let $a = \mathfrak{Re}(\beta_1)$, $b = \mathfrak{Im}(\beta_1)$, $P = \mathfrak{Re}(K_1)$, and $Q = \mathfrak{Im}(K_1)$. Then from (5.41), we have

$$
\begin{aligned}
D &= \beta_2 \sinh(A_1)\cosh(A_2) - \beta_1 \cosh(A_1)\sinh(A_2) \\
&= \beta_1^* \sinh(A_1)\cosh(A_1^*) - \beta_1 \cosh(A_1)\sinh(A_1^*) \\
&= 2\,\mathfrak{Im}\left(\beta_1^* \sinh(A_1)\cosh(A_1^*)\right) \\
&= a\sin(2(bx + Q)) - b\sinh(2(ax + P)).
\end{aligned}
$$

Therefore D changes sign as x goes from large negative values to large positive values, and hence D must equal zero for some $x \in \mathbf{R}$. Therefore, as remarked above, $u(\beta_1, \beta_2, x)$ cannot be continued to an analytic function on $\mathbf{R}$.

Next, suppose β_1^2 and β_2^2 are real and of opposite sign, say $\beta_1^2 < 0 < \beta_2^2$. In this case β_1 is purely imaginary, say $\beta_1 = ib$, and β_2 is real. Then when M_1 and M_2 are chosen so that (5.43) holds, it follows that $(e^{2M_1})^* = e^{-2M_1}$ and $(e^{2M_2})^* = e^{2M_2}$. We can thus take M_1 to be purely imaginary, and M_2 so that either M_2 is real or $\mathfrak{Im}\, M_2 = \pi/2$. Now the same arguments as in equations (4.20) to (4.21) in Section 4 show that

$$y_1 = i\sin(bx + K_1) \tag{5.51}$$

for some real K_1, and either

$$y_2 = \sinh(\beta_2 x + K_2) \tag{5.52}$$

or

$$y_2 = i\cosh(\beta_2 x + K_2) \tag{5.53}$$

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for some real K_2. If (5.51) and (5.52) hold, the equation $D = 0$ becomes

$$\frac{\tan(bx + K_1)}{b} = \frac{\tanh(\beta_2 x + K_2)}{\beta_2},$$

which has infinitely many solutions on $\mathbf{R}$. If, on the other hand, (5.51) and (5.53) hold, then the equation $D = 0$ becomes

$$\frac{\tan(bx + K_1)}{b} = \frac{\coth(\beta_2 x + K_2)}{\beta_2},$$

which again has infinitely many solutions.

Now suppose that β_1^2 and β_2^2 are both negative, say $\beta_1^2 < \beta_2^2 < 0$. Then β_1 and β_2 are both purely imaginary, say $\beta_j = ib_j$ for $j = 1, 2$, and in (5.43) we can choose M_1 and M_2 to be purely imaginary. Thus

$$y_j = i \sin(b_j x + K_j)$$

on I for $j = 1, 2$, where K_1 and K_2 are real, so D has a zero at any point $x \in \mathbf{R}$ which satisfies the equation

$$\frac{\tan(b_1 x + K_1)}{b_1} = \frac{\tan(b_2 x + K_2)}{b_2}. \tag{5.54}$$

Since $\beta_1^2 \neq \beta_2^2$, then $b_1 \neq b_2$, in which case it is easy to see that equation (5.54) always has solutions.

We have now shown that, under the stated assumptions on β_1 and β_2, $u(\beta_1, \beta_2, x)$ can be continued to an analytic function on $\mathbf{R}$ only if β_1^2 and β_2^2 are both positive, with therefore $0 < \beta_1 < \beta_2$. In this case, when M_1 and M_2 are chosen to satisfy (5.43), we will have that $(e^{2M_j})^* = e^{2M_j}$ for $j = 1, 2$. Then, as in (5.52) and (5.53), for each j we have that either

$$y_j = \sinh(\beta_j x + K_j)$$

or

$$y_j = i \cosh(\beta_j x + K_j),$$

where K_1 and K_2 are real. There are therefore four cases to consider, which, according to Definition 3.6, can be summarized as follows:

$$u = \psi^{(2)}(x; a_1, a_2; \beta_1, \beta_2),$$

where a_j are nonzero numbers given by $a_j = \pm e^{-2K_j}$ for $j = 1, 2$.

If a_1 and a_2 are both positive, then $D = 0$ holds at any point x where

$$\beta_2 \tanh(\beta_2 x + K_2) - \beta_1 \tanh(\beta_1 x + K_1) = 0, \tag{5.55}$$

and since the left-hand side of (5.55) changes sign from negative to positive as x increases from large negative values to large positive values, there must exist

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solutions to (5.55) in $\mathbf{R}$. Similarly, if a_1 and a_2 are both negative, then $D = 0$ when

$$\beta_1 \tanh(\beta_2 x + K_2) - \beta_2 \tanh(\beta_1 x + K_1) = 0,$$

which again must have at least one solution in $\mathbf{R}$. If $a_1 < 0 < a_2$, then $D = 0$ when

$$\tanh(\beta_1 x + K_1) \tanh(\beta_2 x + K_2) = \frac{\beta_1}{\beta_2}$$

which must have a solution, since the fraction on the right-hand side is between 0 and 1, and the function on the left-hand side attains the value zero at the points $x = -K_j/\beta_j$, $j = 1, 2$, and approaches 1 as $x \to \pm\infty$.

Finally, if $a_2 < 0 < a_1$, then $D = 0$ only if

$$\tanh(\beta_1 x + K_1) \tanh(\beta_2 x + K_2) = \frac{\beta_2}{\beta_1}. \tag{5.56}$$

But (5.56) has no solutions, since the fraction on the right-hand side is greater than 1, and the function on the left-hand side always takes values less than 1. (Alternatively, in this final case we could deduce from Lemma 3.10 that D is never 0 on $\mathbf{R}$.) $\qquad\square$

5.1   The nondegenerate case

In this subsection we consider the case in which the roots C_1 and C_2 of (5.5) are nondegenerate: that is, when $C_1 \neq C_2$, $C_1 \neq 0$, and $C_2 \neq 0$. Recall that, as mentioned following equation (5.14), we can assume that either C_1 and C_2 are both real, with $C_1 < C_2$, or $C_1^* = C_2$. From our definition of α_j in (5.27) and the remarks following, we have then that α_1 and α_2 are distinct and both nonzero, with $0 \leq \mathfrak{Re}\,\alpha_1 \leq \mathfrak{Re}\,\alpha_2$; and either α_1^2 and α_2^2 are both real, with $\alpha_1^2 < \alpha_2^2$, or $\alpha_1^* = \alpha_2$.

According to Corollary 5.4, $u(x) = u(\alpha_1, \alpha_2, x)$ on some neighborhood of x_0, and therefore $u(\alpha_1, \alpha_2, x)$ can be analytically continued to a function which is analytic on all of $\mathbf{R}$. It then follows from Lemma 5.6 that $0 < \alpha_1 < \alpha_2$, and

$$u(x) = \psi^{(2)}(x; a_1, a_2; \alpha_1, \alpha_2)$$

for some numbers a_1 and a_2 with $0 < a_1 < a_2$.

We have therefore proved Theorem 5.2 in the nondegenerate case, by showing that the only possible solutions in this case are given by (5.6).

To complete the proof of Theorem 5.2, it remains to show that the degenerate cases, when C_1 and C_2 can coincide or vanish, cannot arise under the assumption that (5.1) has a nontrivial solution $u(x)$ in $H^2(\mathbf{R})$. Since any H^2 solution must be analytic on $\mathbf{R}$, to accomplish this it is enough to show that in the degenerate cases, any locally analytic solution of (5.1) extends analytically to a function with a singularity on $\mathbf{R}$.

5.2 The degenerate case $C_1 = C_2 \neq 0$

In this subsection we consider the case when equation (5.5) has a nonzero double root, so that $C_1 = C_2 = C \neq 0$ in (5.31). In this case, C must be real. Define α to be a square root of C, following the convention for choice of square roots set after (5.27). From Corollary 5.4 we have that

$$u(x) = u(\alpha, \alpha, x) = \lim_{\epsilon \to 0} u(\alpha, \alpha + \epsilon, x) \qquad (5.57)$$

for $|x - x_0| < \delta_1$.

We now compute the limit in (5.57). Since $\alpha \neq 0$, by taking ϵ positive and sufficiently small, we may assume that $\beta_1 = \alpha$ and $\beta_2 = \alpha + \epsilon$ satisfy the hypotheses of Lemma 5.5. We thus obtain that, for $|x - x_0| < \delta_1$,

$$u(\alpha, \alpha + \epsilon, x) = 2(D(\alpha, \alpha + \epsilon, x)'/D(\alpha, \alpha + \epsilon, x))' \qquad (5.58)$$

where for all complex numbers s and t we define $D(s, t, x)$ by

$$D(s, t, x) = t \sinh A(s, x) \cosh A(t, x) - s \sinh A(t, x) \cosh A(s, x), \qquad (5.59)$$

with

$$A(s, x) = s(x - x_0) + M(s)$$

and

$$e^{M(s)} = \frac{(s + V_1)(s + V_2)}{(s - V_1)(s - V_2)}. \qquad (5.60)$$

Since the right side of (5.60) is nonzero for $s = \alpha$, we may assume that $M(s)$ is defined and analytic for s in some neighborhood of α.

Observe that for $\epsilon > 0$ we can rewrite equation (5.58) in the form

$$u(\alpha, \alpha + \epsilon, x) = 2(D_1(\epsilon, x)'/D_1(\epsilon, x))', \qquad (5.61)$$

where

$$D_1(\epsilon, x) = \frac{D(\alpha, \alpha + \epsilon, x)}{\epsilon} = \frac{D(\alpha, \alpha + \epsilon, x) - D(\alpha, \alpha, x)}{\epsilon}.$$

Let us define

$$D_1(0, x) = \lim_{\epsilon \to 0} D_1(\epsilon, x) = \left.\frac{\partial}{\partial \epsilon}\right|_{\epsilon=0} D(\alpha, \alpha + \epsilon, x).$$

Computing the derivative gives

$$D_1(0, x) = \frac{1}{2} \sinh(2\alpha(x - x_0) + 2M(\alpha)) - \alpha(x - x_0) - \alpha M'(\alpha) \qquad (5.62)$$

for all $x \in \mathbf{R}$. Choose I_1 to be any nonempty subinterval of $(x_0 - \delta_1, x_0 + \delta_1)$ such that $D_1(0, x) \neq 0$ on the closure of I_1. Then, as ϵ goes to 0, $D_1(\epsilon, x)$ will converge to $D_1(0, x)$ uniformly on I_1, and the derivatives with respect to x of

$D_1(\epsilon, x)$ will converge to the corresponding derivatives of $D_1(0, x)$ uniformly on I_1 as well. It then follows from (5.57) and (5.61) that

$$u(x) = 2(D_1(0, x)'/D_1(0, x))' \tag{5.63}$$

for $x \in I_1$.

We now show that (5.63) extends analytically to a singular function on $\mathbf{R}$. For this it is enough to show that $D_1(0, x)$ has at least one zero on $\mathbf{R}$.

If C is positive, then α is a positive real number. Recalling that either V_1 and V_2 are both real or $V_1^* = V_2$, we see that $e^{M(\alpha)}$ is real. Depending on whether the right-hand side of (5.60) is positive or negative at $s = \alpha$, we can choose $M(\alpha)$ to either be real, or to have imaginary part $\pi/2$. Also since (5.60) implies

$$M'(s) = \frac{(s + V_1) + (s + V_2)}{(s + V_1)(s + V_2)} - \frac{(s - V_1) + (s - V_2)}{(s - V_1)(s - V_2)}, \tag{5.64}$$

$M'(\alpha)$ is real in any case.

If $M(\alpha)$ is real, from (5.62) we see that $D_1(0, x) > 0$ for x large and positive, and $D_1(0, x) < 0$ for x large and negative. Therefore there must exist at least one $x \in \mathbf{R}$ for which $D_1(0, x) = 0$. On the other hand, if $M(\alpha) = K + i\pi/2$ for K real, then (5.62) gives

$$D_1(0, x) = -\frac{1}{2} \sinh(2\alpha(x - x_0) + 2K) - \alpha(x - x_0) - \alpha M'(\alpha),$$

and again we see that $D_1(0, x) = 0$ for some $x \in \mathbf{R}$.

If C is negative, then $\alpha = i\gamma$ for some positive number γ. We see from (5.60) that $e^{M(i\gamma)}$ has modulus 1, and so $M(i\gamma)$ can be taken to be purely imaginary: say $M(i\gamma) = i\eta_1$ for $\eta_1 \in \mathbf{R}$. From (5.64) it is readily checked that $M'(i\gamma)^* = M'(i\gamma)$, and hence $M'(i\gamma)$ is real: say $M'(i\gamma) = \eta_2$ for $\eta_2 \in \mathbf{R}$. Then from (5.62) we have that

$$D_1(0, x) = i\left(\frac{1}{2} \sin(2\gamma(x - x_0) + 2\eta_1) - \gamma(x - x_0) - \gamma\eta_2\right). \tag{5.65}$$

The quantity in parentheses in (5.65) is a real-valued function of x which is positive for large negative values of x, and is negative for large positive values of x. Therefore $D_1(0, x)$ must equal zero for some $x \in \mathbf{R}$.

We have shown, then, that whenever $C_1 = C_2 \neq 0$, $u(x)$ is given on some open interval by the function on the right hand side of (5.63), which however cannot be extended analytically to a function on all of $\mathbf{R}$. This contradicts the fact that $u(x)$ is analytic on $\mathbf{R}$. Thus no nontrivial H^2 solutions to (5.1) can exist in this case.

5.3 The degenerate case when $C_1 = 0$ or $C_2 = 0$ (but not both)

Here we consider the cases when either $0 = C_1 < C_2$ or $C_1 < C_2 = 0$.

Suppose first that $C_1 = 0$ and $C_2 > 0$, and let $\alpha = \sqrt{C_2}$. From Corollary 5.39 we have that

$$u(x) = u(0, \alpha, x) = \lim_{\epsilon \to 0} u(\epsilon, \alpha, x) \qquad (5.66)$$

for $|x - x_0| < \delta_1$. For ϵ sufficiently small, we can take $\beta_1 = \epsilon$ and $\beta_2 = \alpha$ in Lemma 5.5, obtaining

$$u(\epsilon, \alpha, x) = 2(D(\epsilon, \alpha, x)'/D(\epsilon, \alpha, x))', \qquad (5.67)$$

where D is as defined in (5.59). Again, we may assume that $M(s)$ in (5.60) is defined and analytic for s in some neighborhood of α and for s in some neighborhood of 0. Note that we can take $M(0) = 0$, so $A(0, x) = 0$.

As in (5.61), we can write, for all $\epsilon > 0$,

$$u(\epsilon, \alpha, x) = 2(D_2(\epsilon, x)'/D_2(\epsilon, x))', \qquad (5.68)$$

where

$$D_2(\epsilon, x) = \frac{D(\epsilon, \alpha, x)}{\epsilon} = \frac{D(\epsilon, \alpha, x) - D(0, \alpha, x)}{\epsilon}$$

and $D(0, \alpha, x) = 0$. Define

$$D_2(0, x) = \lim_{\epsilon \to 0} D_2(\epsilon, x) = \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} D(\epsilon, \alpha, x).$$

We find by differentiating that

$$D_2(0, x) = \alpha \cosh(\alpha(x - x_0) + M(\alpha))\,[x - x_0 + M'(0)] - \sinh(\alpha(x - x_0) + M(\alpha)). \qquad (5.69)$$

As in Subsection 5.2, on any subinterval of $\{|x - x_0| < \delta_1\}$ where $D_2(0, x)$ is bounded away from zero, it follows from (5.66) and (5.68) that

$$u(x) = u(0, \alpha, x) = 2(D_2(0, x)'/D_2(0, x))'. \qquad (5.70)$$

Now $D_2(0, x) = 0$ at any point x for which

$$\alpha(x - x_0 + M'(0)) = \tanh(\alpha(x - x_0) + M(\alpha)). \qquad (5.71)$$

Again as in Subsection 5.2, since α is real then $M'(\alpha)$ is real and $M(\alpha)$ can be taken to either be real or to have imaginary part $\pi/2$. Since $\tanh(x + i\pi/2) = \tanh(x)$ for all $x \in \mathbf{R}$, in either case (5.71) takes the form

$$\alpha x - \beta_1 = \tanh(\alpha x - \beta_2),$$

where β_1 and β_2 are real numbers, and hence must have a solution at some point in $\mathbf{R}$. It follows then from (5.70) that u cannot be analytically continued to all of $\mathbf{R}$.

There remains to consider the case when $C_1 < 0$ and $C_2 = 0$. Let $\alpha = i\beta$, where $\beta > 0$ and $\beta^2 = |C_1|$. From Corollary 5.39 we have

$$u(x) = u(\alpha, 0, x) = \lim_{\epsilon \to 0} u(\alpha, \epsilon, x),$$

and Lemma (5.5) gives, for ϵ sufficiently small,

$$u(\alpha, \epsilon, x) = 2(D(\alpha, \epsilon, x)'/D(\alpha, \epsilon, x))' = 2(D(\epsilon, \alpha, x)'/D(\epsilon, \alpha, x))',$$

provided $|x - x_0| < \delta_1$, where D is still given by (5.59). Hence $u(\alpha, \epsilon, x)$ is still given by the right-hand side of (5.67), and the calculations in (5.68) to (5.71) apply to $u(\alpha, \epsilon, x)$, the only difference being that now $\alpha = i\beta$ is purely imaginary. In this case, as seen in Subsection 5.2, $M(\alpha)$ can be taken to be purely imaginary, and so (5.71) can be rewritten in the form

$$\beta x - \beta_1 = \tan(\beta x - \beta_2),$$

where β_1 and β_2 are real numbers. This equation has (infinitely many) real solutions, and so again u cannot be analytically continued to all of $\mathbf{R}$.

We have thus proved that when either C_1 or C_2 (but not both) is zero, then u cannot be analytically continued to $\mathbf{R}$. So under the assumption that u is an H^2 solution of (5.1), this case cannot arise.

5.4   The degenerate case $C_1 = C_2 = 0$

Finally we consider the case when C_1 and C_2 are both zero. Then, by Corollary (5.4),

$$u(x) = u(0, 0, x) = \lim_{\alpha \to 0} u(0, \alpha, x), \tag{5.72}$$

where $u(0, \alpha, x)$ is given for $\alpha > 0$ by (5.70) with (5.69). To emphasize the dependence of $D_2(0, x)$ on α, let us denote $D_2(0, x)$ by $D_3(\alpha, x)$ in what follows. That is,

$$D_3(\alpha, x) = \alpha \cosh(\alpha(x - x_0) + M(\alpha))[x - x_0 + M'(0)] - \sinh(\alpha(x - x_0) + M(\alpha)).$$

The limit in (5.72) is more singular than those in preceding sections, because $D_3(\alpha, x)$ has a zero of order three at $\alpha = 0$. That is, we have

$$D_3(0, x) = \frac{\partial D_3}{\partial \alpha}(0, x) = \frac{\partial^2 D_3}{\partial \alpha^2}(0, x) = 0.$$

Therefore, to obtain a formula for $u(0, 0, x)$, we should define

$$D_4(\alpha, x) = \frac{D_3(\alpha, x)}{\alpha^3},$$

and we will have

$$u(0, 0, x) = 2(D_4(0, x)'/D_4(0, x))',$$

where

$$D_4(0, x) = \lim_{\alpha \to 0} D_4(\alpha, x) = \frac{1}{6} \left. \frac{\partial^3}{\partial \alpha^3} \right|_{\alpha = 0} D_3(\alpha, x). \tag{5.73}$$

An elementary but fairly tedious computation of the derivative in (5.73) shows that

$$D_4(0, x) = \frac{1}{3}(x - x_0 + M'(0))^3 - \frac{1}{6}M'''(0),$$

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and as $M'(s)$ is real for all real s by (5.64), we have that $M'(0)$ and $M'''(0)$ are real. Clearly then $D_4(0, x)$ has a zero at some $x \in \mathbf{R}$, and so $u(x) = u(0, 0, x)$ cannot be extended to an analytic function on $\mathbf{R}$.

To summarize, we have now shown that in all the degenerate cases, when C_1 or C_2 are zero or when $C_1 = C_2$, (5.1) cannot have an H^2 solution on $\mathbf{R}$; and in the nondegenerate case the only possible solutions are given by (5.6). This then completes the proof of Theorem 5.2. $\qquad\square$

6 The stationary equation for general N

We conclude with a few comments as to how the results above may be generalized to arbitary stationary equations of the KdV hierarchy. In view of the first remark following Definition 3.12, it is natural to conjecture the following generalization of Theorems 4.2 and 5.2: if $u \in H^{2N-2}$ is a nontrivial distribution solution of the stationary equation

$$d_3 R_3 + d_5 R_5 + d_7 R_7 + \cdots + d_{2N+3} R_{2N+3} = 0,$$

then u must be a k-soliton profile for the KdV hierarchy, for some $k \in \{1, 2, \ldots, N\}$. More precisely, there must exist real numbers α_j and a_j, with $(-1)^{j-1} a_j > 0$ for $j = 1, \ldots, k$, such that

$$u(x) = \psi^{(k)}(x; a_1, \ldots, a_k; \alpha_1, \ldots, \alpha_k).$$

Much of the proof given above for Theorem 5.2 generalizes immediately to arbitrary N. The extension of Lemma 5.1 to arbitrary N, with H^2 replaced by H^{2N-2}, is obvious. Let $C_1, \ldots C_N$ be the roots of the equation

$$d_{2N+3} z^N - d_{2N+1} z^{N-1} + d_{2N-1} z^{N-2} - \cdots \pm d_3, \qquad (6.1)$$

and let α_j be the square roots of the C_j, suitably defined. The generalization to arbitary N of the definition of $\hat{R}(x, \zeta)$ is already given in [7], along with the proof that in the case when $\hat{R}(x, \zeta)$ has distinct roots $\zeta_1, \ldots, \zeta_N$ at some x_0, they satisfy an analogue of the system (5.22).

Using induction and an argument like that given above in Section 5, we can assume that the ζ_j are in fact distinct, since otherwise (5.22) reduces to a system with a smaller value of N. From (5.22) one then obtains a generalization of the system (5.31) for functions v_j which are suitably defined square roots of the functions $-\zeta_j$. Lemma 5.3, Corollary 5.4, and Lemma 5.5 all generalize straightforwardly to arbitrary N. We thus obtain that any solution $u(x)$ of (2.15) is given by

$$u(x) = u(\alpha_1, \ldots, \alpha_n, x) = \lim_{\beta_1 \to \alpha_1, \ldots, \beta_N \to \alpha_N} \psi^{(N)}(x; a_1, \ldots, a_N; \beta_1, \ldots, \beta_N),$$
$$(6.2)$$

for some complex numbers $a_1, \ldots, a_N$, where the limit is taken through values of $(\beta_1, \ldots, \beta_N)$ such that the β_j are distinct and all non-zero.

To complete the proof of the conjectured general result, then, it would remain to do two things. First, establish an analogue of Lemma 5.6, or in other words establish the conjecture mentioned in the first remark after Definition 3.12; and second, show that in the degenerate cases when some of the C_j coincide or are equal to zero, the functions $\psi^{(N)}(x; a_1, \ldots, a_N; \beta_1, \ldots, \beta_N)$ in (6.2) converge to a limit which cannot be analytically continued to all of $\mathbf{R}$.

7   Acknowledgements

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